I Equations

I.1 Introduction

Preliminary remarks: As suggested by Rudolf Steiner, a real study of equations begins in the seventh grade: "And, above all, try to bring them into what, in connection with free usage in practical life, is referred to as the study of equations." It will be brought up again in the eighth grade.[37] At that time there will be some things to add that were not dealt with in the seventh grade, and other things can be elaborated. This elaboration on the theme of equations can be taken in different directions: Algebraically, the demands can become more challenging. This applies, above all, to the necessary conversions in formulating solutions. Then, the so-called story problems can become more relevant to practical life. Finally, one can think of simple equation systems with two unknown factors. However, every teacher must determine their own limitations and those of the class.

We begin the study of equations with *ratios* and *proportions*. We may also connect this with interest rate calculations as it is presented in the volume *Mathematikunterricht in der sechsten Klasse an Waldorfschulen.* If conversion of the interest formula has been studied in the sixth grade one can make the connection with a few practice exercises. It is also just as possible to go immediately into ratios. For those of whom the whole area of ratios and proportions seems impenetrable, the easier introduction may be chosen as is described starting on page 181.

Some of what leads up to the understanding of ratios has been presented in the book *Der Anfangsunterricht in der Mathematik an Waldorfschulen.* There, it is explained how with ratios we are no longer dealing with the original sizes themselves, but rather with their *relationships*. Since many teachers may no longer be so familiar with ratios and proportions I will include some things that may be helpful. Those who are able to get into it with their class will be contributing much to a form of free thinking that is unencumbered by conceptions of size. This can also contribute to an enlivening of natural-scientific thinking.

[37] See also chapter 5 in Ernst Bindel's *Die Arithmetik. Menschenkundliche Begruendung und paedagogische bedeutung*, Stuttgart 1967.

I.2 Ratios and Proportions

A basic activity of human thinking is the understanding of relationships between what is perceived, imagined, or thought. We can look at the relationship between two colors, two tones, two numbers, or two concepts as that of cause and effect. Relationships are different than the content itself. They stand, so to say, on a higher level. They are, in themselves, generally not the same as that which is related by succession. The relationship of two colors is not another color, but rather a *color nuance*, and the relationship of two tones is not another tone, but rather an *interval*.

A special case of relationships is quantity equations: If two things of equal quantities are given, we can compare them in different ways. The two easiest possibilities are the question of their *difference* and the question of their *relationship*.[38] The Swiss mathematician, Louis Locher-Ernst (1906 – 1962) beautifully pointed out the difference between these two ways of comparison.[39] The relationships to human soul forces – thinking, feeling, willing – that he characterized can be understood if one imagines two boards of different lengths, for example. Their *difference* can be materially represented by a piece of board, but yet still be very close to the world of perception or imagination. Their *ratio* cannot be understood in the same material fashion. Understanding it requires a much stronger inner *activity*, whose substance, however, is constantly withdrawing itself from perception. If perception wishes to understand the ratio itself, the only thing left are the parts that are in relationship to one another. The substance is experienced much more in the *process*, the measured comparison.

At first, ratios are experienced independent of numbers. For instance, if we experience the pleasant, harmonious ratios of length, width, and height of a beautiful building, in its entirety, or only in single aspects, the numbers involved are not immediately perceived but their relationship is *felt*, i.e. perceived through feeling. If we find a common measure for two lengths, or other similar quantities, we can express their relationship through numbers. Through measuring one can find number ratios that, in perceiving, would be unconsciously felt. So, all music touches on felt ratios.[40] In general, proportions are understood often as correlations of changeable quantities, such as the countering movement of both ends of a two-armed lever, but also between length and force, and many other situations.

I.3 What does Measuring Mean?

Already in the third grade there was a lot of measuring done: lengths in cubits, spans, steps, or meters, volumes in liters, areas in acres, hectares, or square meters, etc. All measuring is a *comparison*. But one can only compare something that is the same in a specific aspect. Quantities like length, area, volume, and many more, are measured by comparing them with another length, area, or volume. Therefore, one chooses a unit of measure to which all the other quantities are related. To measure a length with another length means to tell how many times the measuring length is contained in the other. For example, if one has a 12.5 meter long piece of string and measures it with a 2.5 meter long string, then the measurement is 5. The 5 tells how many times the measuring length goes into the measured length: $12.5\text{m} = 5 \cdot 2.5\text{m}$.

I.4 Measuring and Dividing

For a more complete understanding of ratios and proportions it is useful to try and get clarity about the relationships between measuring and dividing, between ratio formation and division.[41] A differentiation is only possible when the commutative principle of multiplication

[38] We will not be looking at logarithms here.
[39] Louis Locher-Ernst, *Mathematik als Vorschule zur Geist-Erkenntnis,* Dornach 1973.
[40] See also the chapter titled “From the Physics of Music”.
[41] See Louis Locher-Ernst, *Arithmetik und Algebra,* p. 404, same author, *Mathematik als Vorschule zur Geist-*

is interpreted as in the chapter on basic rules and principles of algebra.

It says that when forming a product, the result does not depend upon the order of the factors; the factors may be exchanged; expressed algebraically: $a \cdot b = b \cdot a$. If one relates the calculation to a quantity such as a length, for example, then it is a big difference if four 6-meter beams or six 4-meter beams are available. The total length in both cases remains 24 meters, but for their practical use it is essential to know which factor is the multiplier and which is the multiplicand. The = sign tells us what the equivalency is in regards to the total amount, but it does not tell us about the form of the structure.

If one understands the multiplier as the active *a*, and the multiplicand as the passive *p*, and designates the result as *r*, then the product has this form:

$$a \cdot p = r$$

There are *two* conversion operations that come out of this. There can be *r* and *a* one time, and *r* and *p* one time, and inquiry about the third:

Erkenntnis, Ernst Schuberth, *Der Anfangsunterricht in der Mathematik an Waldorfschulen,* Stuttgart 2001. See also chapter titled “The Basic Rules and Properties of Algebra”.

Multiplication

Measure or Ratio Construction		Divide or Division
	$a \cdot p = r$	
$r : p = a$		$\frac{r}{a} = p$

With the use of named quantities the difference becomes immediately apparent:

If $4 \cdot 6m = 24m$, then *measuring*, or *ratio construction*, means determining the ratio number 4:

$$24m : 6m = 4.$$

As an unnamed number, the 4 is not itself a length, but rather expresses a *relationship* in the way described above. It cannot be understood as a beam length, for instance, like the other named quantities.

The division

$$\frac{24m}{4} = 6m$$

leads again to a named quantity, that is, to the length 6m.

When leaving material quantities, one should contemplate the reason why ratios and proportions play such a prominent role in art.

If one only pays attention to the formal structure of measuring and dividing, then, because of the commutative principle, the two conversion operations need not be differentiated. For this reason all principles of algebra and arithmetic apply to both operations in the same way. However, for the sake of clarity, it must be emphasized that formal, structural equality does not logically mean identical. It is much more important to determine which aspect is more significant, form or content.

I.5 Ratios

Preparation for studying ratios was already begun in the first grade, and then especially in the fourth and fifth grades.[42] First of all, we should prepare some exercises that review estimating ratios. This happens in two directions:

- Two lengths or two other quantities of the same kind are shown and the students are questioned about the ratio number.

- One asks the students to divide a given length, or another quantity, into a specific ratio.

Once the understanding of these questions has been reawakened, the study of ratio calculations can begin in a more systematic way.

If two lengths are given that are in a certain ratio to each other, such as 12m: 8m, for example, then we can express this ratio using larger or smaller numbers:

$$12m:8m = 24m:16m = 6m:4m = 3m:2m = 3:2$$

The value $3:2 = 1.5$ that all these ratios have in common, indicates the *ratio number*, the *ratio factor*, the *value of the ratio,* or also the *measured value*.

In general, in a ratio $a:b$, the a is designated as the first term, and b is the second. If we have

[42] cf. Ernst Schuberth, Volume I (*Der Anfangsunterricht in der Mathematik an Waldorfschulen)* and Volume II of this series (Volume II is not yet available).

a: b = m, then we use the above designation for m. The a and b must always be the same kind of quantities such as pure numbers, lengths, volumes, speed, and the like. The ratio term is always a *pure* number. That is why the ratios for different quantities can be the same, which plays an important role in calculating proportions.

Since, in form, ratios can be dealt with the same as fractions, one can also expand and shorten ratios without changing their value.[43]

If both terms of a ratio are multiplied by the same number (expanded) or divided by the same number (shortened), the value of the ratio remains unchanged.

$a:b = (n \cdot a):(n \cdot b)$

Example:

$12:8 = 24:16 = 3:2$

If $a:b = m$, then we call $b:a = 1/m$ the *reciprocal* or the *inverse ratio*. If we have multiple numbers or same-kind quantities put into ratios, then we speak of *chain ratios* or a *ratio chain*.

Practice 43

1. Determine the value of the following ratios and express them with the smallest possible whole numbers and decimal fractions:

a) *144m : 12m* b) *10kg : 8kg* c) *256 : (-32)* d) $9\tfrac{1}{3}mm : 7mm$ e) *11,431 : 1,6l*

f) $-37 : 2{,}96$ g) $18\tfrac{1}{3} : 2\tfrac{1}{5}$ h) *27,9sec : 3,1sec* i) *333m : 222m* j) *37°: 8°*

Solutions:

a) *12 : 1=12*; b) *5 : 4 = 1,25*; c) *–8*; d) $4:3 = 1{,}\overline{3}$; e) *71 : 10 = 7,1*; f) *-25 : 2 = -12,5*; g) $25:3 = 8{,}\overline{3}$; h) *9 : 1 = 9*; i) *3 : 2 = 1,5*; j) *37 : 8 = 4,625.*

2. Convert the following ratios so that the first factor is 6:

a) *2 : 3* b) *3 : 7* c) *-12 : 18* d) *0,5 : 11* e) *1,5 : 6* f) *4 : 3* g) *8 : 5* h) *9 : (-12)*

Solutions:

a) 6 : 9; b) 6 : 14; c) 6 : (-9); d) 6 : 132; e) 6 : 24; f) 6 : 4,5; g) 6 : 3,75; h) 6 : (-8).

3. The second factor should be 100 (%!):

a) *7 : 10* b) *13 : 20* c) *14 : 25* d) *7,5 : 50* e) *120 : 200* f) *0,5 : 4* g) $\tfrac{1}{2} : \tfrac{1}{10}$

h) $\tfrac{1}{3} : \tfrac{1}{4}$

Solutions:

a) 70 : 100; b) 65 : 100; c) 56 : 100; d) 15 : 100; e) 60 : 100; f) 12,5 : 100; g) 500 : 100; h) $4:3 = \frac{400}{3}:100 = 133{,}\overline{3}:100$.

4. Put in the numbers u = 8 and v = 6 and determine the following ratios:

a) $u : v$ b) $(u + v) : v$ c) $(v - u) : u$ d) $(u + v) : (u - v)$ e) $(u \cdot u) : (v \cdot v)$

f) $(u + v) : (u \cdot u + v \cdot v)$ g) $(u \cdot u - 2v \cdot v) : (u + v)$

[43] This is discussed in Volume II of this series.

Solutions:

a) 4 : 3; b) 7 : 3; c) –1 : 4; d) 7 : 1; e) 16 : 9; f) 7 : 50; g) –4 : 7.

5. Divide the number 100 into a ratio:

a) *1 : 1* b) *1 : 2* c) *2 : 3* d) *3 : 4* e) *3 : 5* f) *3 : 7*

Solutions:

If 100 is to be divided into a ratio a:b, in addition, 100 must be divided into a + b divisions. *One* such division has the quantity x = 100 : (a + b).

It is then, 100 = ax + bx.

For each problem we get:

a) x = 100 : (1 + 1) = 100 : 2 = 50. a = b = 1. Then, 100 = 1 · 50 + 1 · 50.

In the same way, for the rest we get:

b) $x = 100 : (1+3) = \frac{100}{3}$. Dann ist $100 = 1 \cdot \frac{100}{3} + 2 \cdot \frac{100}{3} \approx 33{,}33 + 66{,}67$

c) $100 = 2 \cdot 20 + 3 \cdot 20 = 40 + 60$

d) $100 = 3 \cdot \frac{100}{7} + 4 \cdot \frac{100}{7} = \frac{300}{7} + \frac{400}{7} \approx 42{,}86 + 57{,}14$

e) $100 = 3 \cdot \frac{100}{8} + 5 \cdot \frac{100}{8} = 3 \cdot 12{,}5 + 5 \cdot 12{,}5 = 37{,}5 + 62{,}5$

f) $100 = 3 \cdot 10 + 7 \cdot 10 = 30 + 70$

6. Divide the number 675 into a ratio of 10:17.

Solution: 675 = 10 · 25 + 17 · 25 = 250 + 425

7. The same for 861 into a ratio of 8:13.

Solution: 861 = 8 · 41 + 13 · 41 = 328 + 533

8. Represent the number 72 as the difference of two numbers a and b that behave like 5:9.

Solution:

It should be 72 = b – a and a:b = 5:9. We divide 72 into b – a = 4 equal parts: 72:4 = 18. a = 5 · 18 = 90 and b = 9 · 18 = 162 to fulfill the requirements.

9. Do the same for 165 and a:b = 12:17.

Solution: 165 : (12 – 7) = 165 : 5 = 33. 165 = 12 · 33 – 7 · 33.

10. Divide the number 750 into a ratio of 2:3:5.

Solution:

750 is divided into 2 + 3 + 5 = 10 equal parts x. Then is 750 = 2 · x + 3 · x + 5 · x = 150 + 225 + 375, and it is 150 : 225 : 375 = 2 : 3 : 5.

11. Do the same for 840 and 3 : 4 : 5.

Solution: 840 : (3 + 4 + 5) = 840 : 12 = 70. 840 = 3 · 70 + 4 · 70 + 5 · 70 = 210 + 280 + 350 and 210 : 280 : 350 = 3 : 4 : 5.

12. The volume ratio of oxygen to nitrogen in the air is 21 : 79. How much oxygen and how much nitrogen are in 1250m^3 of air?

Solution:

1250 : (21 + 79) = 1250 : 100 = 12.5. Then is 21 · 12.5m^3 = 262.5m^3 the amount of oxygen and 79 · 12.5m^3 = 987.5m^3 the amount of nitrogen.

13. Many maps represent distance by scale. In the following table, determine each missing quantity. (The *solutions* are given in *brackets*. Take into account: 1km = 100,000cm.)

Scale	Distance on the Map in cm	Actual Distance in km
1 : 20.000	7,0	*1,4*
1 : 20.000	*9,5*	1,9
1 : 50.000	*12,0*	6,0
1 : 500.000	*20,0*	100,0
1 : 500.000	*6,0*	30,0
1 : 2.000.000	14,0	*280,0*
1 : 2.500.000	13,4	*335,0*

14. Three sisters, ages 14, 12, and 9, want to carry home a load (white rocks that they gathered for the garden, for instance) that weighs approximately 70kg. They decide to divide up the load according to their age. How much would each one have to carry if they could measure it exactly on a scale?

Solution:

First, we divide the 70kg into 9 + 12 + 14 = 35 parts: 70kg : 35 = 2kg.

Each one now gets a load based on her age. That is:

14 · 2kg = 28kg, 12 · 2kg = 24kg und 9 · 2kg = 18kg.

Together, these loads will equal 70 kg.

15. A business owner is going bankrupt. After auctioning off the bankruptcy assets and deducting the expenses of the bankruptcy proceedings there are $30,000 left. Several vendors have outstanding invoices for the business: A $9,500, B $15,000, C $20,000, and D $5,500. How much will each creditor receive if the money is divided according to the ratios of the outstanding claims?

Solution:

This can be approached in different ways. Using the example of problem 14, we could divide the $30,000 by the total sum of the claims in the amounts of $9,500 + $15,000 + $20,000 + $5,500 = $50,000. In that way we get how much can be paid out per $1.00. If we multiply this by the claim amount of each creditor, we get the amount to be paid out. $30,000 : $50,000 = 0.6. That is to say, every dollar of the claim is worth only $0.60 to the creditor. A gets $9.500,- × 0,6 = $5.700,-, B $15.000,- × 0,6 = $9.000,-, C $20.000,- × 0,6 = $12.000,-, D $5.500,- × 0,6 = $3.300,-. Together, these amounts are $5.700,- + $9.000,- + $12.000,- + $3.300,- = $30.000,-.

16. Solve problem 15 using the following numbers: Sale of the bankruptcy assets total $20,000. Claims from creditors: A $120,000, B $60,000, C $40,000, D $20,000.

Solution: Here the losses of the creditors are much greater: $1.00 is worth only 1/12th of its value. The creditors receive: A $10,000, B $5,000, C $3,333.33 D $1,666.67.

17. A neglected house has stood unused for years. Three architects get it at an auction for $90,000 and they each pay $30,000. Architect A puts up $50,000 for the renovation, architect B $90,000, and architect C $60,000. After the renovation is complete the house is sold for $400,000. How should the profit be divided?

Solution: The total costs amount to $290,000. The profit is $400,000 - $290,000 = $110,000. It should be divided according to the amount each person paid toward the purchase and renovation. The total amount paid was: A \$80,000, B \$120,000, C \$90,000. For each dollar put in, there was \$110,000 : \$290,000 = 0.379… paid out. This gives for A \$80,000 · 0.379 = \$30,340, B \$120,000 · 0.379 = \$45,520, and C \$90,000 · 0.379 = \$34,140.

18. If there had been a loss instead of a profit in problem 17 in the amount of \$58,000, how would that be divided among the architects?

Solution:

For each dollar put in \$0.20 is forfeited. This gives for A \$80,000 · 0.2 = \$16,000, B \$120,000 · 0.2 = \$24,000, C \$90,000 · 0.2 = \$18,000.

The last problem and more problems of a similar nature will be looked at once again in a later chapter (see page 172 and following page).

Proportions

If the ratios

$$a : b \text{ and } c : d$$

are equal, then one describes the equality

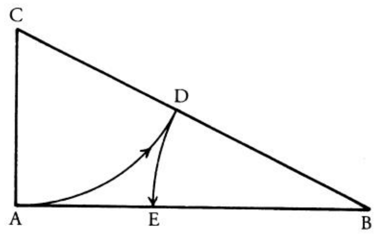
$$a : b = c : d$$

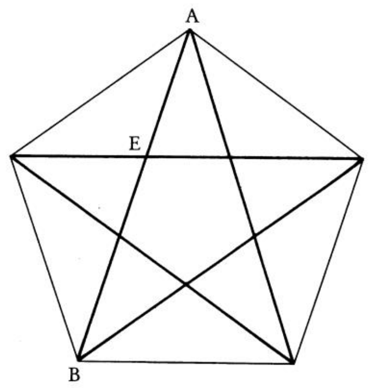
as *proportion*. One says: *a is to b as c to d.* The quantities a and c are called the *antecedents* of the proportion, and b and d are the *consequents*; a and d are the *extreme* terms and b and c are the *inner terms*, **or means**. **(Trans. Note: some dictionaries call the inner terms the means)** One calls d the fourth proportional to the quantities a, b, and c, whereby the order of sequence is very important.

If the inner terms of a proportion are equal, then one speaks of a *continued proportion*.

If: a : b = b : d, then b is the *mean proportional* to the quantities a and d, or also the *geometric mean* to the quantities a and d. If a segment of the length d is uniformly divided, one says it is divided by the *golden ratio*.

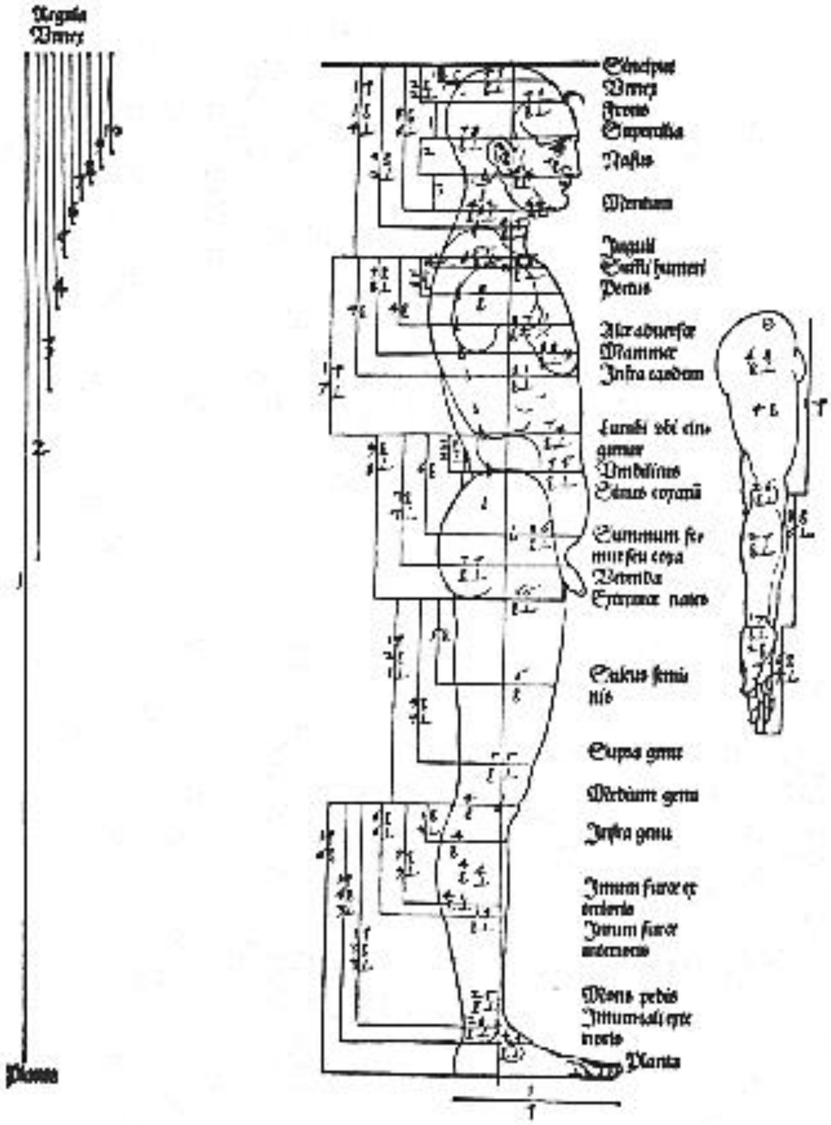
For example, in a regular pentagon the leg b is the mean proportional between the diagonals d = AB **(Trans. Note: equations are shown at bottom of page 160)** and the oddment d – b = AE. The diagonals alternately divide each other in proportion with the golden ratio. The drawing on the left shows how, based on the length d = AB, this segment can be divided in proportion to the golden ratio.



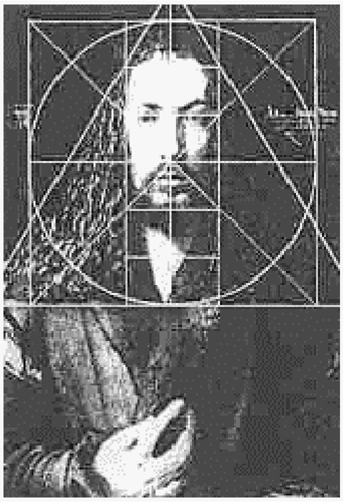


Graphic 3: The construction of the golden ratio and the golden ratio in the pentagram

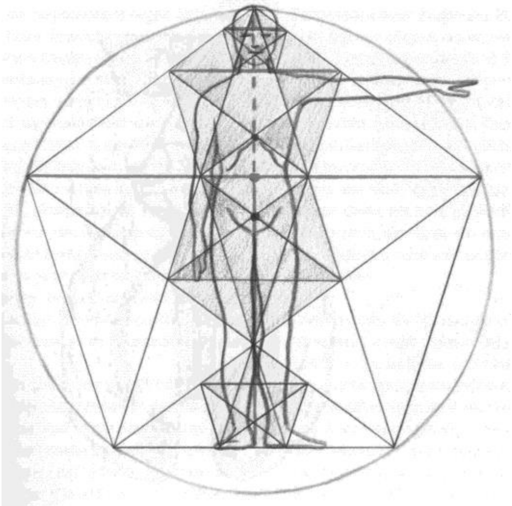
In many ways, the human body is formed according to the proportion of the golden ratio.[44]



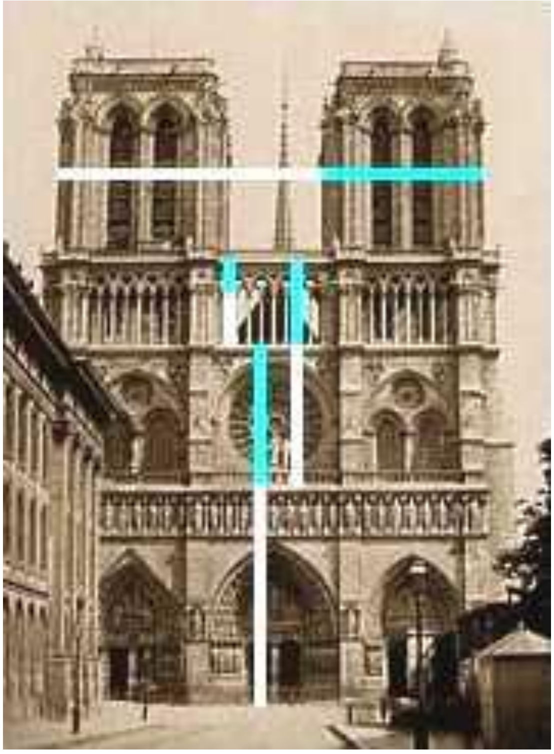
[44] See for example Walter Buehler, *Das Pentagramm und er Goldene Schnitt als Schoepfungsprinzip,* Stuttgart 2001; Duerers sketch: *Der Goldene Schnitt am Koerper,* Johannes W. Rohen, Formprinzipien im menschlichen Koerperbau und das Raumerleben des Menschen. (Formative principles in the human body and the human spatial experience). In: *Mensch und Architektur,* Nr. 42/43. 10/2003, P. 16 – 23.



Graphic 5: Albrecht Duerer, Self Portrait with the Golden Ratio



Drawing 6: Johannes W. Rohen, The Golden Ratio and the Human Body



Graphic 7: The Golden Ratio and the Notre Dame Cathedral

If a : b = c : d, then we can multiply or divide both sides with the same number. If we do this with the product of the consequents b · d, then we get:

$a \cdot d = c \cdot b$

This also applies:

(1) A proportion is only correct when the product of the extreme terms equals the product of the mean terms:

a : b = c : d applies exactly if also $a \cdot d = b \cdot c$

The rule regarding the conversion of a proportion into a product equation is often expressed like this:

extreme term · extreme term = inner term · inner term ***(Trans. Note: inner term could also be mean term)***

It is the most important way of converting a proportion.

This also applies:

(2) Other correct proportions come from the proportion a : b = c : d if one exchanges either the extreme terms with each other, the mean terms with each other, or a mean term with the appropriate extreme term.

If one has a proportion between the numbers or quantities a, b, c, d, then there are eight different basic forms present, namely:

$a : b = c : d$	$b : a = d : c$	$c : a = d : b$	$d : b = c : a$
$a : c = b : d$	$b : d = a : c$	$c : d = a : b$	$d : c = b : a$

The easiest way to check this is by converting all the proportions into products. One always gets:

$a \cdot d = b \cdot c$ *or rather,* $b \cdot c = a \cdot d$.

For proportions, this also applies:

(3) The sum or difference of the terms on the left side of a proportion is to one of these terms the same as the sum or difference of the terms on the right side.

From a : b = c : d follows the proportions

$$(a \pm b) : a = (c \pm d) : c \text{ and } (a \pm b) : b = (c \pm d) : d$$

The accuracy follows through multiplying it out. Further proportions can be derived by using (2).

The same goes for multiple sums of terms of a continued proportion, which will be discussed further below.

This rule can be genially practiced using the relationships between two numbers a and b, and their greatest common divisors, ggT(a,b), and their smallest common multiples, kgV(a,b).[45] This applies:

$$a \cdot b = ggT\,(a,b) \cdot kgV\,(a,b)$$

If one arranges the numbers into a square according to their greatest common dividers and their smallest common multiples, then it is easy to follow all the changes. Let us choose, for example, the numbers 8 and 12. Then we have ggT (8, 12) = 4 and kgV (8,12) = 24. Arranged in a square, we have:

$$\begin{array}{ccc} & 24 & \\ 8 & & 12 \\ & 4 & \end{array}$$

These are the proportions: 24 : 8 = 12: 4 and 24 : 12 = 8 : 4. However, according to rule (3) we also have: (24 $\pm$ 8) : 24 = (12 $\pm$ 4) : 12, then 32 : 24 = 16 : 12 = 4 : 3 or rather 16 : 24 = 8 : 12 = 2 : 3.

Practice 44

1. Measure your height and divide it by the golden ratio. Where do the corresponding points lie (measured once from above and once from below)? Divide the largest segments again and look for corresponding lengths on your own body.

2. Determine ggT (72,120) with the help of proportions that apply in the formula given above. Tip: First determine kgV (72,120).

3. Determine ggT (231, 363) from the two given numbers, and from kgV (231, 363) = 2541.

4. What is the number for x in kgV (x, 156) = 780 and ggT (x, 156) = 13?

Solutions:

1. Compare the graphics on page 161 and following page; 2. kgV (72, 120) = 24; ggt (72, 120) = 360; 3. kgV (231, 363) = 33; 4. x = 65.

Reciprocals and Compound Ratios

If two quantities, a and b, are given, then their ratio can be expressed in two ways: One way is, after looking at the ratio a : b, one looks at it the other way around, b : a. For example, if someone is 60 years old and another person is 20 years old, then the older person will have an age ratio of 60 : 20 = 3 : 1. The older person is three times the age of the younger person. On the other hand, the younger person has an age ratio of 20 : 60 = 1 : 3. The younger person is 1/3rd the age of the older person.

(4) The ratios a : b and b : a are called reciprocals of each other. Their product is 1, and this applies: The ratio of two numbers is equal to the reciprocal ratio of the reciprocal

[45] Cf. Volume II of this series.

numbers.

a : b = (1 : b) : (1 : a)

This double view of a ratio is often of special significance. In this way square ratios $a^2 : b^2$ can also be understood as ratios of the two ratios a : b and b : a.[46]

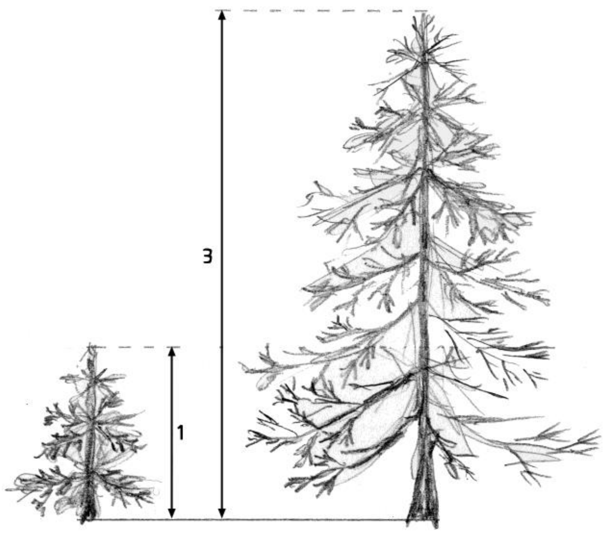
Examples:

1. If a = 4m, b = 3m, then (a : b) : (b : a) = (4m : 3m) : (3m : 4m) = 16 : 9
2. The Pythagorean Theorem can also be expressed in this form:

(a : c) : (c : a) + (b : c) : (c : a) = 1

It becomes then a rule about proportions.

In life it is often helpful to be able to see things from the point of view of both sides! We can once again clarify both points of view by using the ratio of two heights.



Graphic 8: The Height Ratio 1 : 3 or 3 : 1

We find an important example of a reciprocal ratio with the mechanical law of the lever. Through experimentation we can more specifically determine what we already instinctively know through our own experience:

In order for a two-armed lever to remain in balance both loads L_1 and L_2 on each end must be the reverse of the corresponding arms of the lever, l_1 and l_2.

$L_1 : L_2 = l_2 : l_1$.

For example, if a father who weighs 70kg wants to teeter totter with his two daughters who weigh 30kg together, on a board that is 5m long, then the board must be supported at a 1.5m distance from the father:

30kg : 70kg = 1.5m : 3.5m.

Practice 45

1. Determine which of the following cases have to do with direct ratios, indirect ratios, or neither:

a) Expenses and quantity of goods (as long as there are no volume discounts)

b) Deployment of laborers and working hours for a specific task

[46] Johannes Kepler, in his work *Harmonics Mundi*, devoted long passages to the subject of simple and square proportions.

c) Speed and roadway (in equal time)
d) Size and age of a person
e) Length of shadow and height of an object (in equal sun position)
f) Use of gasoline and automobile expenses
g) Speed and driving time of an automobile for a given distance

Solutions:
a) direct; b) indirect; c) direct; d) neither; e) direct; f) neither; g) indirect.

From the Physics of Music

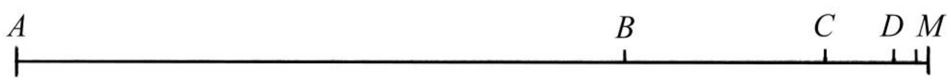
The significance of ratios that are equal, i.e. proportions, can be experienced especially well in music. A musical interval – prime, second, third, fourth, fifth, sixth, seventh, octave, etc. – requires from the string instruments that their sounding strings be either lengthened or shortened. The pitch depends upon the absolute length (besides tension of the string and other factors). The ratio of an interval is characteristic in that both sounding strings have a ratio distance from one another.

The ratio 3 : 2 is always characteristic of a fifth interval, the ratio 4 : 3 a fourth interval, and so on. What we describe as a ratio in mathematical terms is a feeling experience of tonal sound known as intervals in music. When one tunes a violin or a cello, the first thing one does is change the tension of the A-string and tune it to a certain pitch. After the other strings have been tuned in a fifth interval, one begins to grip the strings in such a way that the ratios that are characteristic for each interval will sound. For example, if one wants to play several fourth intervals in succession, one must first divide the given base length in a ratio of 4 : 3, and again the remaining length at a ratio of 4 : 3, and so forth. Of course, depressing the string on the finger board also raises its tension a little, so that actually, one's finger must always add a little length. It is a wonder how an accomplished violinist can bring their finger movements and experience of tone into perfect balance. For the mind, it is really not easy to completely grasp what the fingers and the sense of hearing are able to accomplish.

The following graphic shows, from left to right, where the equal divisions should go so that the same tone intervals always occur.

**AM : *BM = *BM : *CM = *CM : *DM = ... = 3 : 2 (Fifth Interval).*

This involves *steady* proportions.



(Graphic 9: Division of a string in fifth intervals)

Practice 46

1. One string has the length 1 = 120cm. Find the length x that will allow the base tone of a fourth interval to sound.

Solution:

It must be: x : 120cm = 3 : 4. Converted, it is: $x = \frac{3}{4} \cdot 120cm$ = 90cm.

2. A string 120cm long sounds on C. To what length l should the string be shortened so that the other notes of the C-major or C-minor can sound?

Tip: Purely harmonious tuning is built on the basis of whole number ratios. These are given in the following table for the intervals of the C-major and C-minor scales.

Interval	Tone	Major/Minor	Ratio to Corresponding String Length
Base Tone	c		$1/1$
Second	d		$8/9$
Minor Third	es	Minor	$5/6$
Major Terz	e	Majorr	$4/5$
Forth	f		$3/4$
Fifth	g		$2/3$
Minor Sixth	as	Minor	$5/8$
Major Sixth	a	Major	$3/5$
Minor Seventh	b	Minor	$5/9$
Major Seventh	h	Major	$8/15$
Oktave	c'		$1/2$

Solution: l(d)= $\frac{8}{9}\cdot 120cm = 106{,}67cm$, l(e)= $\frac{4}{5}\cdot 120cm = 96cm$, l(f)=90cm, l(g)=80cm, l(a)=72cm, l(h)=64cm, l(c')=60cm.

Continued Proportions

The ratios a :b, b : c, c : d…are thought of as *connected* because, beginning with the second ratio, the antecedent term is the same as the consequent term of the preceding ratio. Any number of connected ratios can be reproduced in the form of a *continued ratio*:

$a : b : c : d : ...$

If there are multiple proportions in which the ratios to the left and right are connected, then we call the proportions themselves *connected.* For example, if we have:

$a : b = p : q,\ b : c = q : r,\ c : d = r : s,$

one can represent these connected proportions in the form of a *continued proportion*:

$a : b : c : d = p : q : r : s$

Such an expression, in which the same number of terms must be to the left and right, is, initially, only a small demonstration of the individual proportions, but it is accorded certain advantages. The individual terms on one side of a continued proportion are created from the corresponding terms on the other side through multiplying out with a fixed number *w*.

$a : b : c : d = p : q : r : s$

becomes

$a = w \cdot p,\ b = w \cdot q,\ c = w \cdot r,\ d = w \cdot s$

Reversed, the 8 given numbers, a – s, remain in the given ratios and, in that way, can be put in the form of a continued proportion. The two terms on one side of a continued proportion behave the same as the two corresponding terms on the other side. A continued proportion also remains correct if one multiplies or divides corresponding terms on the left and right by the same number.

Examples for Continued Proportions

The following six tones' frequencies have a continued ratio:

1 : 2 : 3 : 4 : 5 : 6

Going consecutively *upward*, they form the intervals of octave, fifth, fourth, major third, and minor third; that is, the main tones of a *major scale* (for example, C, C', G'. C", E", G").

The following six tones' frequencies have a continued ratio:

$$1 : \frac{1}{2} : \frac{1}{3} : \frac{1}{4} : \frac{1}{5} : \frac{1}{6} = 60 : 30 : 20 : 15 : 12 : 10$$

Going consecutively *downward*, they form the intervals of octave, fifth, fourth, major third, and minor third; that is, the main elements of a *minor scale* (for example, c", c', f, c, As, F).

The frequencies of three tones, which form a basic triad in major (prime, major third, fifth), when precisely tuned, have the continued ratio

4 : 5 : 6.

The ratio for the basic triad in minor (prime, minor third, fifth) is

$$\frac{1}{6} : \frac{1}{5} : \frac{1}{4} = 10 : 12 : 15$$

Three consecutive major triads (for example, f-a-c', c'-e'-g', g'h'd") contain the exact seven tones that form a minor scale, though in part offset by an octave. In order to express the frequencies as a continued ratio we must make the ratio

4 : 5 : 6, 4 : 5 : 6, 4 : 5 : 6

connected. Starting with the first two: Expanding by 2, or rather 3, gives:

8 : 10 : 12, 12 : 15 : 18.

Expanding this again by 2, and the 3rd ratio by 9 gives:

16 : 20 : 24 : 30 : 36 : 45 : 54.[47]

Generally, in physics as in music, proportions often occur as natural expressions of laws. So, for instance, the law of falling bodies in a vacuum of space can be illustrated as a proportion: During freefall of an object, the speed corresponds equally to the time: $v_1 : v_2 = t_1 : t_2$. *Kepler* found that the cubes of the principle axes of two planet orbits corresponded to the ratio of the periods of revolution to their reciprocal ratio; that is, $a^3 : a_1^3 = (t : t_1) : (t_1 : t)$.

Practice 47

First, we will return to a previous problem and solve it by way of a continued proportion (see page 158).

1. Three sisters, ages 14, 12, and 9, want to carry home a load (white rocks that they gathered for the garden, for instance) that weighs approximately 70kg. They decide to divide up the load according to their age. How much would each one have to carry if they could measure it exactly on a scale?

Solution:

If we designate the loads of the three children in order as a, b, and c, then it must be that a : b : c = 14 : 12 : 9. From this we can get the individual proportions:

[47] From Louis Locher-Ernst, *Arithmetik und Algebra*, Chapter 29.

a : b = 14 : 12 = 7 : 6 and a : c = 14 : 9.From this we get $b = \frac{6}{7}a$ *and* $c = \frac{9}{14}a$.

Besides that, a + b + c = 70kg. Then, we have:

a + b + c = a + 6/7a + 9/14a = 70kg → 14a + 12a + 9a = 14 · 70kg → 35a =

14 · 70kg → a = 14 · 2kg → a = 28kg.

From this we get the other loads of 24kg and 18kg. The arrow (→) is used to symbolize "it follows".

2. A business owner is going bankrupt. After auctioning off the bankruptcy assets and deducting the expenses of the bankruptcy proceedings there are $30,000 left. Several vendors have outstanding invoices for the business: A $9,500, B $15,000, C $20,000, and D $5,500. How much will each creditor receive if the money is divided according to the ratios of the outstanding claims?

Solution:

If we designate the amounts to be paid as a, b, c, and d, then it must be that:

a : b : c : d = 9500 : 15000 : 20000 : 5500, or shortened:

a : b : c : d = 19 : 30 : 40 : 11. From this we get the individual proportions

a : b = 19 : 30 oder $b = \frac{30}{19}a$ *; a : c =19 : 40* oder $c = \frac{40}{19}a$ *; a : d = 19 : 11* or $d = \frac{11}{19}a$

Because a + b + c + d = $30,000, we get:

A + 30/19a + 40/19a + 11/19a = 100/19a = $30,000, → a = $5,700.

From the individual proportions we get the other values:

b = $9,000; c = $12,000; d = $3,000.

Together these amounts are:

$5,700 + $9,000 + $12,000 + $3,300 = $30,000.

3. Solve problems 16, 17, and 18 from page 159 in the same way.

4. Pyrite consists of about 4 parts by weight sulfur and 7 parts iron. How much sulfur and iron are in 300g of pyrite?

Solution:

4 + 7 = 11; 300 : 11 ≈ 27,27g; 4 · 27,27.g =109,09 (sulfur); 7 · 27,27.g = 190,91.g (iron).

Compound Ratios

To close this chapter we will once again practice problems in which the quantities to be found depend upon multiple other changeable quantities. Besides the standard assignments, these should be given as special problems for homework. In the next lesson time slot, thoughts about solutions should be gathered. It is important that the assignment be calmly and comprehensively discussed to the satisfaction of all. It is not the amount of problems that increases skill, but the careful discussion of a few. It can also be very motivating to let the class work in small groups and give them problems with varying degrees of difficulty. Many times those less skilled in arithmetic prove themselves, under these conditions, to be very good problem solvers!

1. A group of three girls decides to make *fifteen* calendars with *twelve* geometric drawings each for the school bazaar. There is three weeks time left. They decide to meet *three* times per week and work. After *one* week they have finished only *three* calendars. They ask *two* friends to help them and they also decide to meet *four* times per week. Will they finish the

fifteen calendars as planned?

Solution:

We will go about it step by step. Per meeting the three of them were able to finish one calendar, that is, one-third of a calendar per person (four drawings). If they were five people instead of three and assuming they all were equally productive, they would complete 5 · 1/3 calendars per meeting (= 20 drawings). In the last two weeks they will meet eight times. Mathematically, they could complete

8 · 5 · 1/3 calendars = 40 · 1/3 calendars = 13 1/3 calendars in that time. If they make 13 more calendars then they will exceed their goal because they have already made 3 calendars in the first week, so, together they will have 16 calendars.

Now, let us go through the calculations once again and systematically write down the unknown values as well as those we are looking for. On the top line we write the known quantities from the first situation and underneath, the corresponding quantities in the changed situation:

3 Girls	1 Week	3 Days per Week	3 Calendars
5 Girls	2 Weeks	4 Days per Week	x Calendars

The number x of the calendars to be made directly depends upon every quantity. For instance, the ratio of the unknown number x to the 3 calendars made in one week by three girls is the same as the ratio of the work days per week:

$$x : 3 = 4 : 3 \text{ or rather } \quad x = 3\cdot\frac{4}{3}.$$

And it goes the same for the other ratios. We put x in direct proportion to the ratios in which the individual quantities change:

$$x = 3\cdot\frac{5}{3}\cdot\frac{2}{1}\cdot\frac{4}{3} = \frac{40}{3} = 13\frac{1}{3}.$$

Then we can carry out the further considerations.

If among the quantities there had been one from which the number of calendars was dependent upon that quantity in reverse, we would have to put in the reciprocal ratio at the appropriate place.

Example: Three girls make three calendars in a three-day work week. How much time would five girls need to make 15 calendars in a four-day work week?

Now, it is a question of *time*, and the time *lessens* with the number of girls and the number of work days per week. This is what the calculation would be:

3 Girls	*1 Week*	*3 Days per Week*	*3 Calendars*
5 Girls	*x Weeks*	*4 Days per Week*	*15 Calendars*

$$x = 1\cdot\frac{3}{5}\cdot\frac{3}{4}\cdot\frac{15}{3} = \frac{9}{4} = 2\frac{1}{4} \text{ weeks = 2 weeks and 1 day (for a 4-day week).}$$

2. High water is threatening to overflow a dike. It must be raised along a 4.2km distance. The flood surge is expected in 36 hours. 30 people are able to adequately stack sand bags along a 50m stretch of dike per hour. How many people are necessary to get the dike sandbagged before the flood surge?

Solution:

30 People	*1 Hour*	*50m*
x People	*36 Hours*	*4200m*

The number of people necessary is directly proportional to the distance and is reciprocally proportional to the time. This applies:

$$x = 30 \cdot \frac{1}{36} \cdot \frac{4200}{50} = 70\, People$$

But, we have not yet taken into consideration that the people need to eat and rest. In the direst of circumstances, and without further help, they could probably do it without sleep and food. There are enough examples of this!

3. A ship with a 38-person crew has 76kg of meat on board. This gives 0.2kg of meat per person per day for the ten-day cruise. After the ship has left harbor, two blind stowaways are discovered. The captain is a good-hearted man and allows them the same share of rations as the crew, but he reduces the meat ration to 190g. Will there be enough meat?

Solution:

38 People	*10 Days*	*0.20kg*
40 People	*x Days*	*0.19kg*

The number of days the meat will last is reciprocally proportional to the number of people and the quantity of the ration. Therefore, this applies:

$$x = 10 \cdot \frac{38}{40} \cdot \frac{0{,}20}{0{,}19} = 10$$

The meat will last for 10 days.

Another type of problem has to do with filling and inflation times and related issues. Again, the following assignment should first be given to the students as a problem:

5. A swimming pool must be filled in the spring. There are *two* filling hoses. Alone, the first hose can fill the pool in 10 hours, and the second in 15 hours. How long would it take to fill the pool if both hoses were on?

Solution:

Again, there are different things to be considered that lead to the same result. For instance, one can think about the following: The first problem is that the filling time may not be added. Can you find a quantity that may be added? One thing to look at is the flow. That is the amount of water that flows in a certain period of time such as hours or minutes. If V is the volume (water content of pool), then the flow of the first hose is S_1 = V/10h, and the flow of the second hose is S_2 = V/15h. The flow of both hoses is:

$$S = S_1 + S_2 = \frac{V}{10h} + \frac{V}{15h} = V \cdot \frac{25}{150h} = \frac{V}{6h}$$

The volume V of the pool will flow out of both hoses in 6h.

If the filling time of the first hose is t_1 and the second hose is t_2, then the filling time for both hoses can be calculated like this:

$$\frac{1}{t} = \frac{1}{t_1} + \frac{1}{t_2} \quad or \quad t = \frac{t_1 \cdot t_2}{t_1 + t_2}$$

Another type of problem plays a role where one has to convert from one system of measurement into another, such as from the European metric system to the American system. It has to do with compound ratios.

6. On average, a car uses 6.7 liters of gasoline for 100km. How many gallons (U.S.) is that for 100 miles?

Solution:

We are presenting this problem in order to show a somewhat different method of writing that is advantageous in certain cases. The related quantities are written next to each other, but we begin with the unknown number x of gallons per 100 miles and continue so that all measured quantities occur, and the last kind of quantity on the right is repeated on the left side underneath. In this way it is certain that x is given in the right unit of measure.

x l	100 Meilen
1 Meile	1,61 km
100 km	6,7 l
1 l	0,264 Gallonen (US)

We get the unknown x by multiplying all the quantities on the right side and dividing by all the quantities on the left (without x):

$$x = \frac{100 \cdot 1{,}61 \cdot 6{,}7 \cdot 0{,}264}{100} \approx 2{,}85$$

In Europe, if an average usage of gasoline per 100km is given as 6.7 liters, then in the USA that would be 2.85 gallons per 100 miles. However, in the USA it is more often given how many miles per gallon can be driven. This is the distance:

d = 100/2.85 miles = 35 miles

General Equations

After a thorough study of proportions, we turn to the study of general equations. If we want to solve a problem with the help of mathematics then we will make use of *equations* in many cases. We have done this in the previous chapter through proportions. The way we accomplished this, through *conversion,* is also the way we will go about it with general equations. The basic principles that we are already familiar with are important helps.

Before we systematically begin with the simplest type of equations, we will give an assignment that shows a typical use. One can present it to the class and wait and see how they react. If, after persistent consideration, the problem still appears too difficult, the solution can be put on hold and revisited later.[48]

Problem:

The Situation: A man and his partner are hiking in the Rocky Mountains in North America. The woman is bitten in the leg by a snake. The man only got a brief glance of the snake. There are no cell phone signals in the area, so they left the cell phone in the car. They put a tourniquet on her leg as well as they could. The woman cannot possibly walk back. They decide that the man will go back the 6 miles to the nearest ranger station.[49] At the station they have anti-venom serum for all the snakes that are found in the area. Since he is traveling downhill, the man arrives at the station in only one hour. He is not sure what kind of snake it was, so the rangers give him a chart to look at with photographs of the different kinds. Finally, he finds one that he thinks was the snake, although he only got a very short glimpse of it. A ranger starts out immediately with the serum. The man is too tired to go back with the ranger. He is talking with another ranger at the station when suddenly the ranger remembers a snake that has similar markings to the one on the chart but is seldom seen. He shows a picture of it to the man and he is now sure that that is the snake he saw. What to do? The wrong serum could be deadly. The other ranger has already been gone one-half hour. She will be able to make it up the mountain in two hours. The other ranger starts out immediately with the right serum. How fast must he be to catch up with the first ranger?

[48] cf. also from Rudolf Steiner in the 12th seminar discussion in *Erziehungskunst. Seminarbesprechungen und Lehrplanvortraege* (Complete Works 295); examples of problems dealing with movement.

[49] A U.S. mile is about 1.6km.

Preparing the Solution:

The students should work through the solution as independently as possible. It is very interesting to see how different students go about it. As a teacher, one can often learn from this observation. It would be unfortunate to insist upon a specific path to the solution. Especially when new problems are presented, as many students as possible should be able to suggest possible ways of finding the answer. It is also very appropriate to work in groups. Sometimes I have sent a group to another room if their discussion becomes too loud and disturbs the others.[50]

Solution:

One possible way: The first ranger will need about 2 hours which means she is walking an average of 3 miles per hour. She has 1 ½ = 3/2 hours to go. In this amount of time the second ranger must go the entire 6 miles. That is 2 miles per ½ hour, or 4 miles per hour, or 6.4km per hour up the mountain! That is the minimum average speed he must reach.

The solution for general problems:

Is there a general way of solving a typical problem of this kind? How fast must the ranger go on average in order to catch up to the first ranger in B, who had a head start t_0?[51]

Solution (Found on Page 180)

If the first ranger *t* needs $\overline{AB}$ to go the full distance and the head start time in t_0, then the second ranger has $t - t_0$ to go the entire distance. The speed of the first ranger $v_1 = \frac{\overline{AB}}{t}$, and the speed of ranger two $v_2 = \frac{\overline{AB}}{t - t_0}$

We now know the average speed needed by ranger two, v_2.

If one forms a ratio of the speeds, then one gets an immediately apparent principle in the form of a proportion:

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[50] See also Johann Sjuts, different mental constructions for problem solving…In: *Journal for Math* (2002) Issue 2, pp. 106 – 128.

[51] The relationship between distance, time, and speed, should be generally discussed here. Since mechanics are begun in the 7th grade, a small foray into mathematical kinematics, which is usually done only later in Waldorf Schools, can not do any harm. It can be done in various ways. For instance, it is immediately apparent that the distance traveled in a certain period of time is related to speed and time in the following way: Distance = Speed · Time. In physics *s* is normally used to designate distance, *v* for speed (Latin *velocitas),* and *t* for time (Latin tempus), which gives us: s = v · t. Once converting has become familiar, one can also determine the other formulas like this:

$$speed = \frac{distance}{time} \quad or \quad v = \frac{s}{t} \quad and \quad time = \frac{distance}{speed} \quad or \quad t = \frac{s}{v}.$$

$$v_1 : v_2 = \frac{\overline{AB}}{t} : \frac{\overline{AB}}{t - t_0} = (t - t_0) : t$$

The speeds behave the opposite of the times; the greater the speed, the less time needed and vice-versa.

A Use for the Principle:

If someone needs 4 hours to go a distance of 18km and has a head start of 1 hour, then a second person must go the same distance in 3 hours in order to arrive at the same time as the first person. The speed of the first person is:

$v_1 = \frac{18km}{4h} = 4{,}5km/h$ The speed of the second person is: $v_2 = \frac{18km}{3h} = 6km/h$

If we want to use this newfound principle, we could say:

$v_1 : v_2 = (t - t_0) : t.$ *(*)*

We know v_1, t and t_0; and we are trying to find v2, the unknown speed of the second person. This is the proportion from our example:

$4{,}5km/h : v_2 = 3 : 4.$

Applying the rule, mean terms · mean terms = extreme terms · extreme terms, we get:

$v_2 \cdot (t - t_0) = v_1 \cdot t$ *or* $v_2 = v_1 \cdot \frac{t}{t - t_0}.$

In our example it would be:

$$v_2 = \frac{4}{3} \cdot 4{,}5km/h = 4 \cdot 1{,}5km/h = 6km/h.$$

This is example is intended to show how equations can be used.