I Equations – A Systematic Beginning

I.1 Introduction

In this chapter we learn about a more mechanical way of solving equations. Good skills help very much to transform terms and equations. They do *not* explain how in a special case you can find the needed equation, but once you found it, you will be able to find the answer. How to find a correct equation needs a lot of experience and therefore we develop in chapter II of this booklet carefully different examples.

The following simple exercises are much easier than the problems that have already been done, but they afford a good opportunity for students who are weaker in this subject to have a new beginning. Essentially, it has to do with practice in converting equations, and, of course, mathematically formulating and solving practical problems. As far as possible, one should try and impart some life skills when dealing with the problems. The examples serve only as possible suggestions.

A systematic study of equations can begin with mental arithmetic that students probably already started with in the first grade. They can be treated as number riddles or guessing games. The teacher will say something like: I am thinking of a number, and when I add 5 to it, the result is 10. What number am I thinking of? The students must use the result and/or the completed operation to get the unknown number.

Now, this will be brought up again systematically. The following examples show types of problems and how they can be successively dealt with. Above all, they serve to bring the path to problem solving into consciousness. If a class or an individual learner is well-practiced in mental arithmetic, the difficulty of the problems can be raised to an astounding level. Of course, an individual learner will need somebody who gives him or her problems. Such problems are wonderfully suited for the necessary exercising of memory because they require that one retain memory of the mathematical operation and then invert it. Here, where we have usually given three numbers that we call *a (*for *active), p (*for *passive),* and *r (*for *result).* In a class the teacher should initially give concrete numbers! The unknown number is designated by *x*, as is the normal practice.

I.2 A more Systematic Start with Equations

First, we describe the different types of problems that we will start with:

- I am thinking of a number x and add a. The result is r. What is the number?
- A number x is added to a number p. The result is r. What number was added to p?

- I am thinking of a number x and subtract a. The result is r…
- A number x is subtracted from a number p…
- I am thinking of a number x and multiply it by p. The result is r…
- I have multiplied a number x by a number p. The result is r. By what number have I multiplied p?
- I am thinking of a number x and divide it by a…
- I have divided the number p by x…

The following problems contain *two* mathematical operations:

- I am thinking of a number x, multiply it by a and add b. The result is r…
- I thinking of a number x, add a and multiply the result by b. The result is r…

With these and other similar types of problems, we begin with simple, natural numbers, then also with negative numbers, and, finally, with simple fractions. We meet the different skill levels of the students with varying degrees of difficulty and tempo, and patience. It is always possible to increase the level of difficulty by introducing more than two mathematical operations in one problem. The students must then begin with the last operation and work through the calculation in reverse.

It is also appropriate to do these exercises in small groups in which each in turn is given a problem.

After the first exercises, we call attention to the fact that in order to solve the problem one must always do the reverse operation from what the teacher initially said. If there is addition, then one must subtract to find the solution, and vice-versa. We go over again the pairs of opposing operations. Besides this, the inverse operations must be carried out in reverse order.

At the conclusion of the first systematic practice that began with mental arithmetic with increased difficulty over a longer period of time, we will now try to understand the process of finding the solution when *written* on the blackboard.

1. If we have found the unknown number x (other letters are of course possible, and should be used), then added the number 3, for instance, and got the result 7, the written operation would look like this:

$$x + 3 = 7$$

Such an expression is called a *conditional equation.*

How did we find the unknown x? If 7 is the number that was *increased* by 3, then we must *decrease* the number 7 by 3 to get x.

$$x = 7 - 3 = 4$$

And 4 is in fact the solution to the problem because when we check it by putting 4 in place of x in the original equation, we find that both sides are the same value:

$$4 + 3 = 7 \text{ (Check)}$$

2. However, if the number 3 is subtracted from the unknown x in order to get 7, then the written problem looks like this:

$$x - 3 = 7$$

Now, the number x that is decreased by 3 equals 7. We must *increase* 7 by 3 in order to find x:

$$x = 7 + 3 = 10$$

x = 10 is in fact the solution because 10 – 3 = 7 (check)

3. If a number equals 15 when it is tripled, this is the way to find the solution: Since the tripled number is 15 then it is one-third of 15, namely 5.

$3x = 15$

$x = 15 : 3 = 5$

Check: $3 \cdot 5 = 15$

4. If the unknown number divided by 7 equals 4, then we think: If one-seventh of the number is 4, then the number must be seven times 4, namely 28.

$x : 7 = 4$

The solution is:

$x = 4 \cdot 7 = 28$

Check: $28 : 7 = 4$

We can also write out the solution process if there is more than one mathematical operation involved:

5. A doubled number with 3 added to it equals 13:

$2x + 3 = 13$

We find the solution in steps by first reducing 13 by 3 and then dividing by 2:

$2x + 3 = 13, 2x = 13 – 3 = 10, x = 10 : 2 = 5$

If we insert this result into the original equation, we get:

$2 \cdot 5 + 3 = 13$

The unknown number x has been found.

6. The following is solved in the same way:

$2(x + 3) = 16, x + 3 = 16 : 2 = 8, x = 8 – 3 = 5$

To check the answer we substitute 5 for x in the original equation and we get:

$2 \cdot (5 + 3) = 2 \cdot 8 = 16$

7. In the following we proceed in the same way to find x:

$2x – 5 = 7, 2x = 7 + 5 = 12, x = 12 : 2 = 6$

Check: $2 \cdot 6 – 5 = 7$

8. Finally, we solve the following:

$7(x – 2) = 63, x – 2 = 63 : 7 = 9, x = 9 + 2 = 11$

Check: $7(11 – 2) = 7 \cdot 9 = 63$

One should also pay attention to what differences the various brackets make and when brackets can be left out.

Practice 48

The first group can be solved orally or written:

1. I think of a number, add 5 to it, and get 12. What is the number?

2. I think of a number, subtract 7, and get 13…

3. The number 17 is subtracted from another number. 34 remain. What is the other number?

4. I think of a number, multiply it by 9, and get 117…
5. 24 is multiplied by a number x. The result is 360. x =?
6. An unknown number x divided by 13 gives the number 11. x =?
7. 187 divided by a number x gives 11. x =?
8. I think of a number x, multiply it by 12 and add 3. The result is 15. x =?
9. I think of a number x, add 3 and then multiply by 12. The result is 48. x =?
10. The number13 is added to a number x and the result is divided by 7. The answer is 7. x =?
11. A number x is divided by 14. The number 37 is subtracted from the result. The answer is 19. x=?
12. The number 111 is subtracted from a number x. The result is divided by 11. The answer is 0. x =?

Solutions:

1. *7*; 2. *20*; 3. *51*; 4. *13*; 5. *15*; 6. *143*; 7. *17*; 8. *1*; 9. *1*; 10. *36*; 11. *784*; 12. *111*.

Once these types of equations have been adequately practiced, one can begin, in the next days, to discuss basic methods from new viewpoints. Let us take the first example again: If the number x, increased by 3, equals 7, it can be written in this form:

$x + 3 = 7$

From this expression we get:

$x = 7 – 3$ *and* $x = 4$

If, however, it was like this:

$x – 3 = 7$

Then we would get this expression:

$x = 7 + 3$ and $x = 10$

In the solution, with the opposite operation, the 3 appears each time on the right side. In the oral exercises we inverted the chronological order of the operations as well as carried out the opposite of each operation. However, with the written exercises we will look at another viewpoint: It is required of x that it creates equality between the left and right sides of the expression. The conversions needed to fulfill this requirement have been done in steps so that the new equation requires the same number x as the preceding one. To check the answer, we inserted the number result for x that we found into the original equation.

The original meaning of the word algebra, which comes from Arabic, describes the art of creating equations and converting them so that the unknown numbers can be calculated.52

By writing out the calculation, we bring the *chronological* into a *spatial* order. We speak of the *right* and *left* sides of an equation, and with the various conversions we must make sure, as has already been said, that each successive x requirement is equivalent to the preceding one. The balance scale has long been an appropriate image when dealing with equations and their conversions. Both sides of an equation can be compared to the two sides of a balanced

[52] The Persian mathematician Muhammed ibn Musa al-Chwarizmi (algorithms) lived in Bhagdad (ca. 780 – ca. 850). He used the double word Aldschebr walmukabala as the title of one his works about equations. The words mean inversion and reproduction. From there comes the word algebra. He used Indian numerical figures as well as the zero. His work was translated by Leonardo da Pisa Fibonacci (ca. 1170 – ca. 1240). See also: http://www-gap.dcs.st-and.ac.uk/~history/Mathematicians/Al-Khwarizmi.html

scale, and the point of support is the equal sign.

Drawing 11: The Scale as an Image of an Equation

If a scale is in balance one can carry out many operations on both sides without disturbing the balance. We may:

(1) Add the same weight to both sides

(2) Subtract the same weight from both sides

(3) Multiply the weight by the same factor on both sides

(4) Divide the weight by the same factor on both sides

In the first three instances it does not make sense to use the number 0. In the last instance, the use of zero must be expressly forbidden because otherwise inconsistencies could occur since it is true that $0 \cdot 3 = 4 \cdot 0$. But if we divided both sides by 0 we would get $3 = 4$. So, as before, dividing by 0 in this case is not allowed.

This relating to the experience of balance in this way appears to me to be in harmony with the anthropological foundation of mathematics.[53] It is interesting, for instance, that *Luria*, in his book *Working Brain*, pointed out that people who have disturbances in their sense of balance can no longer solve equations.[54]

Corresponding with the experience of balance, one can more generally express the contents of (1) by thinking of *two* scales in balance whose left sides and right sides are added to each other.

Naturally, in mathematics we are not dealing with real weights, but rather with numbers or combined terms. In general, one can say this about the rules: If we have a conditional equation with its left and right sides, then we can get a new equation with the same solutions if we:

(1) Add the same weight to both sides

(2) Subtract the same weight from both sides

(3) Multiply the weight by the same factor on both sides

(4) Divide the weight by the same factor on both sides (0 is not allowed)

But if we want to multiply or divide by more than just one number, say by a term that contains the unknown x, then either more or fewer solutions can occur than in the original equation. The solutions to the original equation must therefore generally not correspond to the newly created equations. For this reason, checking the answers using the beginning equation is so important.

Example:

If the original equation is $x + 2 = 2x - 2$, it has one exact solution; $x = 4$. If we now multiply both sides by $(x - 1)$, we get $(x + 2) \cdot (x - 1) = (2x - 2) \cdot (x - 1)$ with the two solutions of $x_1 = 1$ and $x_2 = 4$. But $x_1 = 1$ is *not* a solution of the original equation. A simple check shows this since 1 + 2 does not = 2 · 1 – 2.

Inversely, if we had begun with the second equation and divided by $(x - 1)$, then we could have easily lost the solution $x_1 = 1$ if we had not thought of the fact that for $x = 1$, the term (x-

[53] cf. Ernst Schuberth, *Der Anfangsunterricht in der Mathematik an Waldorfschulen,* Stuttgart ²2001, Chapter titled: “Rechenschwaechen und die menschenkundlichen Grundlagen der Mathematik”.

[54] Alexander R. Luria, *Working Brain: An Introduction to Neuropsychology,* Basic Books 1973.,

1) would be zero, and we can not divide by zero in this case. If one wants to divide by a term that can equal zero, such as (x – 1), one must first try out the case of (x – 1) = 0, and then divide. In the beginning we will not have such problems as these, but the checks should become a strong habit. Once we have ruled out that the terms by which we are multiplying or dividing are zero, then, through the operations, we will always get equations with the same solutions as the original equation, and, initially, this will always be the case.

The rules (1) – (4) above are very effective tools with which we can, in principle, solve all equations of the kind we have spoken of up to this point. If only one operation is given in an equation, then we use the opposite operation appropriately, as we have already discussed. The applicable operations are those that help to finally isolate the unknown number x (or whatever letter is used) alone on one side of the equation.

Rule (3) is applied especially often with multiplications involving -1, that is, when all prefix signs on both sides of an equation are to be inverted. For example, if one has gotten the expression - x = 4, then one can multiply each side by -1 and get x = -4.

Note: If one wants to commute an equation with a second equation in which both sides are identical, in the manner described above, then one speaks of simply adding or subtracting (or multiplying or dividing) by the same term on both sides of the first equation. In our exercises we will be dealing mostly with numbers in which the same operation can be done on both sides.

Let us go over the already discussed examples once again and practice both methods.

Equations with One Operation

1. Example:

$x + 3 = 7$

In order to isolate the unknown x, we create the – very simple – equation

$-3 = -3$

and add on the appropriate side in each case to get this equation:

$x + 3 – 3 = 7 – 3$ *or*

$x = 4$

Put another way: We subtract the number 3 on both sides of the original equation thereby getting the solution.

Check: $4 + 3 = 7$

2. Example:

$x – 3 = 7$

We form the equation

$3 = 3$

and add to the appropriate sides:

$x – 3 + 3 = 7 + 3;$

and also

$x = 10$

Put another way: We add the number 3 on both sides of the original equation thereby getting the solution.

Check: $10 - 3 = 7$

3. Example:

$3x = 15$

Again, we form the equation

$3 = 3$

But this time we divide the sides by each other:

$$\frac{3x}{3} = \frac{15}{3}$$

That is:

$x = 5$

Put another way: We divide both sides of the original equation by 3 thereby getting the solution.

Check: $3 \cdot 5 = 15$

4. Example:

$$\frac{x}{7} = 4$$

Here, we form the equation

$7 = 7$

Multiply the sides by each other

$$7 \cdot \frac{x}{7} = 7 \cdot 4$$

and get

$x = 28$

In short: We multiply both sides by 7.

Check: $\frac{28}{7} = 4$

Note: The study of equations offers - like many other areas of mathematics – an opportunity to practice basic skills, such as fractions, without making them major themes.

Practice 49

Determine through conversion those numbers that fulfill the established conditions:

	a)	b)	c)	d)	e)	f)
1.	$x + 17 = 34$	$x - 19 = 38$	$x + 122 = 123$	$544 - x = 400$	$543 + x = 729$	$4321 - x = 3210$
2.	$9 + x = 7$	$12 - x = 13$	$15 + x = 7$	$15 - x = 23$	$123 + x = 57$	$519 - x = 1426$
3.	$2764 + x = 2341$	$x - 1{,}5 = 4$	$2{,}7 - x = 5$	$x + 2{,}5 = 6$	$2{,}12 - x = 2{,}11$	$2{,}12 + x = 2{,}11$
4.	$2x = 10$	$5x = 100$	$20x = 80$	$4x = 68$	$17x = 51$	$3x = 75$
5.	$25x = 125$	$5x = 1000$	$200x = 100$	$99x = 33$	$33x = 11$	$11x = 1$
6.	$12x = 6$	$12x = 4$	$12x = 3$	$12x = 2$	$12x = 1$	$12x = \frac{1}{2}$
7.	$12x = \frac{1}{3}$	$0{,}5x = 22$	$0{,}7x = 4{,}2$	$2{,}5x = 62{,}5$	$62{,}5x = 2{,}5$	$7x = \frac{1}{2}$
8.	$\frac{x}{2} = 9$	$\frac{x}{3} = 8$	$\frac{x}{4} = 7$	$\frac{x}{5} = 6$	$\frac{x}{6} = 5$	$\frac{x}{7} = 4$

9. $\frac{x}{8}=3$ $\frac{x}{2}=12$ $\frac{2}{3}x=12$ $\frac{3}{4}x=12$ $\frac{4}{5}x=12$ $\frac{5}{6}x=12$

Solutions:

1. a) *17*; b) *57*; c) *1*; d) *144*; e) *186*; f) *1111*
2. a) *-2*; b) *-1*; c) *-8*; d) *-8*; e) *-66*; f) *-907*
3. a) *-423*; b) *5,5*; c) *–2,3*; d) *3,5*; e) *0,01*; f) *–0,01*
4. a) *5*; b) *20*; c) *4*; d) *17*; e) *3*; f) *25*
5. a) *5*; b) *200*; c) $\frac{1}{2}$; d) $\frac{1}{3}$; e) $\frac{1}{3}$; f) $\frac{1}{11}$
6. a) $\frac{1}{2}$; b) $\frac{1}{3}$; c) $\frac{1}{4}$; d) $\frac{1}{6}$; e) $\frac{1}{12}$; f) $\frac{1}{24}$
7. a) $\frac{1}{36}$; b) *44*; c) *6*; d) *25*; e) $\frac{1}{25}$; f) $\frac{1}{14}$
8. a) *18*; b) *24*; c) *28*; d) *30*; e) *30*; f)*28*
9. a) *24*; b) *24*; c) *18*; d) *16*; e) *15*; f) *14,4*

In general, we can now easily solve all the equations of the following form:

$x + a = b$

We add –a to both sides and get:

$x = b - a$

Check: $b - a + a = b$

Likewise, the solution for x – a = b is x = b + a.

Just as easily, we can, in general solve yyy

$a \cdot x = b$

by dividing both sides by a:

$x = \frac{b}{a}$

Check: $a \cdot \frac{b}{a} = b$

Likewise, the solution for $\frac{x}{a} = b$ is x = a · b

Assignment: Go through the last problems (practice 49) and determine a and b in each case.

Equations with Two Operations

If an equation has *two* operations we must first determine if there is a specific order of the operations because when converting we must reverse the operations.

1. Example:

$2x + 3 = 13$

The operation to be completed last[55]on the left side of the equation is the addition of 3 (*dot before line!).* So, we now subtract the number 3 from both sides:

$2x + 3 - 3 = 13 - 3$

We get:

[55] During the lessons it is important to point out that the "last" operation is not the one that is positioned "last", but rather that the order of operations in this case is determined purely algebraically. The addition would be the last operation if the problem was written as 3 + 2x = 13.

$2x = 10$

Now, we divide both sides by 2 and get:

$2x : 2 = 10 : 2, x = 5$

Check: $2 \cdot 5 + 3 = 13$

2. Example:

$2 \cdot (x + 3) = 16$

Because of the parentheses, the last operation to be completed is multiplying by 2. So first, we divide both sides of the equation by 2:

$2 \cdot (x + 3) : 2 = 16 : 2$

We get:

$x + 3 = 8$

The parentheses can now be left out because it must no longer be made clear in what order the operations should be done. Now, we subtract the number 3 from both sides:

$x + 3 - 3 = 8 - 3$

This gives:

$x = 5$

Check: $2 \cdot (5 + 3) = 16$

We do likewise in the following two examples. We will no longer indicate the single commutations by writing out the expression, but they will be indicated by a vertical slash behind the equation, and to the right, is indicated the operation that is to be done on both sides. The arrow indicates the result.[56]

3rd Example:

$2x - 5 = 7 \mid + 5 \rightarrow 2x = 12 \mid : 2 \rightarrow x = 6$

Check: $2 \cdot 6 - 5 = 12 - 5 = 7$

This mathematical shorthand simplifies the illustration and, once one is used to it, actually makes the calculations clearer.

4. Example:

$7(x - 2) = 63 \mid : 7 \rightarrow x - 2 = 9 \mid + 2 \rightarrow x = 11$

Check: $7(11 - 2) = 7 \cdot 9 = 63$

Here also, we can generally solve all of the important types of these problems:

If the requirement is

$ax + b = c,$

then we first subtract b from both sides and get:

$ax = c - b$

Then we divide both sides by a and get:

$$x = \frac{c - b}{a}$$

[56] Strictly speaking, what we are dealing with here is an equivalent commutation that should be designated by ↔.

Check: $a \cdot \frac{c-b}{a} + b = (c-b) + b = c.$

Using the previous examples, determine which problems have this form: ax + b = c. Determine a, b, and c, and check the answers by using the general solution formula.

One must pay careful attention that the intended commutations are clearly stated – especially in more complicated cases – otherwise the calculation is almost impossible to check.

Practice 50

Calculate the unknown number x by successive commutations and check your answers. With fractions or decimal fractions multiply the equations initially so that the next numbers are whole numbers.

	a)	b)	c)	d)	e)	f)
1.	$2x + 5 = 15$	$2\cdot(x + 5) = 16$	$7x - 12 = 37$	$7\cdot(x - 12) = 35$	$13 - 3x = 16$	$13 - 13x = 0$
2.	$-x + 24 = 25$	$-2x + 47 = 53$	$-3x + 17 = 8$	$-4x + 7 = -21$	$-5x + 10 = -50$	$1 - x = -1$
3.	$5\cdot(2 - x) = -10$	$5\cdot(x - 2) = -10$	$-5\cdot(x - 2) = -10$	$-5\cdot(2 - x) = -10$	$5\cdot(2 - x) = 10$	$-5\cdot(-2 - x) = -10$
4.	$7x = 1$	$7x-2=\frac{3}{2}$	$\frac{3}{2}x+\frac{5}{2}=\frac{23}{2}$	$\frac{1}{3}x+\frac{5}{2}=\frac{9}{2}$	$\frac{3}{4}x+\frac{5}{2}=8$	$\frac{3}{4}x+\frac{5}{6}=\frac{11}{6}$
5.	$0,2x - 2 = 1$	$2x - 0,2 = 1$	$0,7 - x = -0,7$	$0,7 - 0,1x = -0,7$	$0,7 - 0,01x = -0,7$	$0,1x - 0,1 = 0,1$

Solutions:

l.a)5;b)3;c)7;d)17;e)-1;f)1

a) -1; b) -3; c) 9; d) 7; e) 72; f) 2

a) 4; b) 0; c) 4; d) 0; e) 0; f) -4

4.a) $\frac{1}{7}$ *;b)* $\frac{1}{2}$ *;c)6;d)6;e)7;f)* $\frac{2}{9}$

5. a) 15; b) $\frac{3}{20}=0,15$ *; c) 1,4; d) 14; e) 140; f) 2*

Practice 51

Simple word problems are important for translating language into mathematical formulas. Write the conditional equations for the following problems and solve them:

1. If one takes the triple of a number and reduces it by 4, one gets 11. What is the number?
2. If one increases the double of a number by 8, one gets 24.
3. If one adds half of a number to 24, one gets 36.
4. If one reduces half of a number by 3, one gets 9.
5. If one takes away one-third of a number from 654, one gets 111.

Solutions:

1. $3x - 4 = 11, x = 5$; 2. $2x + 8 = 24, x = 8$; 3. $24+\frac{x}{2}=36, x=12;$ 4. $\frac{x}{2}-3=9, x=24;$ 5. $654-\frac{x}{3}=111, x=1629.$

Equations with More than Two Operations

In the previous examples we had the simplest kinds of equations which can be solved in just a few steps. We will now go through a series of examples in which new difficulties appear. The most important one is that the unknown number x now appears in multiple terms:

1. Example:

Calculate x from this equation:

$3x + 4 + 9x = 4x + 19 + 5x$

In order to find x we will combine the terms on each side as far as possible. We get:

$12x + 4 = 9x + 19$

Then we commute the equation so that all the terms with the factor x are on one side – usually the left – and all the pure numbers are on the other side:

$12x + 4 = 9x + 19 \mid -9x \rightarrow 3x + 4 = 19 \mid -4 \rightarrow 3x = 15 \mid : 3 \rightarrow x = 5$

To check the solution we insert 5 for x in the original equation on the left and right sides:

Left Side (LS) $= 3 \cdot 5 + 4 + 9 \cdot 5 = 64$

Right Side (RS) $= 4 \cdot 5 + 19 + 5 \cdot 5 = 64$

x = 5 fulfills the requirements of the original equation.

2. Example:

Find x from the equation

$2x + 3 = x - 9$

We proceed as follows:

$2x + 3 = x - 9 \mid -x \rightarrow x + 3 = -9 \mid -3 \rightarrow x = -12$

Check: $LS = 2 \cdot (-12) + 3 = -21$

$RS = -12 - 9 = -21$

3. Example:

$$\frac{x}{4} + \frac{x}{5} = 9$$

With common denominator:

$$\frac{5x + 4x}{20} = 9$$

Simplify and transform:

$$\frac{9x}{20} = 9 \,\Big|\, :9 \quad \rightarrow \quad \frac{x}{20} = 1 \,\Big|\, \cdot 20 \quad \rightarrow \quad x = 20$$

Check:

$$LS = \frac{20}{4} + \frac{20}{5} = 5 + 4 = 9 = RS$$

4. Example:

$$\frac{6}{x} = 3$$

Here, the x is a denominator in a fraction. So, first of all, we multiply both sides by x. The x then appears on the right side. We may, however, - just like with a balance scale – exchange the two sides of an equation so that we end up with the x on the left side as we are used to having it.

$$\frac{6}{x} = 3 \mid \cdot x \quad \rightarrow \quad 6 = 3x \mid :3 \rightarrow 2 = x \quad \text{or} \quad x = 2$$

Check: LS = $\frac{6}{2} = 3 =$ RS

5. Example:

$$\frac{-2}{x} = \frac{1}{3}$$

Here, also, we first multiply both sides by x, then by 3:

$$\frac{-2}{x} = \frac{1}{3} \mid \cdot x \quad \rightarrow \quad -2 = \frac{x}{3} \mid \cdot 3 \quad \rightarrow \quad -6 = x \quad \text{oder} \quad x = -6$$

Check: $LS = \frac{-2}{-6} = \frac{2}{6} = \frac{1}{3} = RS$

6. Example:

$$1 + \frac{1}{x} = 2$$

Here, we can begin with the sorting:

$$1 + \frac{1}{x} = 2 \mid -1 \quad \rightarrow \quad \frac{1}{x} = 1 \mid \cdot x \quad \rightarrow 1 = x \quad \text{oder} \quad x = 1$$

Check:

$$LS = 1 + \frac{1}{x} = 1 + 1 = 2 = RS$$

7. Example:

$$2 - \frac{5-x}{2} = 1$$

To first get rid of the fraction, multiply both sides by 2.

$$2 - \frac{5-x}{2} = 1 \mid \cdot 2 \quad \rightarrow \quad 4 - (5 - x) = 2$$

Careful: *Every* term is to be multiplied. It is agreed that the fraction line works the same as parentheses. If the line is gone after the multiplication by 2, then, in general, one should put in parentheses, especially if a – sign is in front of the fraction!

Taking away the parentheses gives:

$$4 - 5 + x = 2 \quad \rightarrow \quad -1 + x = 2 \mid +1 \quad \rightarrow \quad x = 3$$

Check:

$$LS = 2 - \frac{5 - 3}{2} = 2 - 1 = 1 = RS$$

Naturally, one can first arrange it like this:

$$2 - \frac{5-x}{2} = 1 \rightarrow \frac{5-x}{2} = 1 \rightarrow 5 - x = 2 \rightarrow x = 3.$$

8. Example:

$$2 - \frac{1}{2-x} = 0$$

Here, one can first arrange it like this:

$$2-\frac{1}{2-x}=0 \quad \Big| +\frac{1}{2-x} \quad \rightarrow \quad 2=\frac{1}{2-x} \quad \Big| \cdot(2-x) \quad \rightarrow \quad 2\cdot(2-x)=1$$

Remove the parentheses:

$4 - 2x = 1 \quad | -4 \quad \rightarrow \quad -2x = -3 \quad | \cdot (-1)$

By multiplying by (-1) the prefix signs on both sides can be reversed:

$2x = 3 \quad | : 2$

$$x=\frac{3}{2}$$

$$\text{Check:} \quad LS = 2 - \frac{1}{2-\frac{3}{2}} = 2 - \frac{1}{\frac{1}{2}} = 2 - 2 = 0 = RS$$

9. Example:

$$\frac{2+x}{4}+\frac{2-x}{2}=0 \quad \Big| \cdot 4 \quad \rightarrow \quad (2+x)+2\cdot(2-x)=0 \quad \rightarrow \quad 2+x+4-2x=0$$

$\rightarrow \quad 6 - x = 0 \quad \rightarrow \quad x = 6$

Check:

$$LS = \frac{2+6}{4}+\frac{2-6}{2}=\frac{8}{4}-\frac{4}{2}=2-2=0=RS$$

The following examples contain more difficult equations and checks:

10th Example:

$$\frac{5x-1}{4}-\frac{1-10x}{5}=\frac{3}{2} \quad \Big| \cdot 20 \quad \rightarrow \quad 5\cdot(5x-1)-4\cdot(1-10x)=30 \quad \rightarrow$$

$$25x-5-4+40x=30 \quad \rightarrow \quad 65x-9=30 \;|\; +9 \quad \rightarrow \quad 65x=39 \;|\; :65 \quad \rightarrow \quad x=\frac{39}{65}=\frac{3}{5}$$

Check: : LS = $\dfrac{5\cdot\frac{3}{5}-1}{4}-\dfrac{1-10\cdot\frac{3}{5}}{5}=\dfrac{3-1}{4}-\dfrac{1-6}{5}=\dfrac{2}{4}-\dfrac{-5}{5}=\dfrac{1}{2}+1=\dfrac{3}{2}=RS$

11. Example:

$$\frac{5x-1}{3}-\frac{1-5x}{4}=7 \quad \Big| \cdot 12 \quad \rightarrow \quad 4(5x-1)-3(1-5x)=84 \quad \rightarrow \quad 20x-4-3+15x=$$

$$84 \quad \rightarrow \quad 35x-7=84 \quad | +7 \quad \rightarrow \quad 35x=91 \quad \rightarrow \quad x=\frac{91}{35} \quad \rightarrow \quad x=\frac{13}{5}$$

Check: $LS = \dfrac{5\cdot\frac{13}{5}-1}{3}-\dfrac{1-5\cdot\frac{13}{5}}{4}=\dfrac{12}{3}-\dfrac{-12}{4}=4+3=7=RS$

12. Example:

$(5 - x)(10 - x) = (4 - x)(13 - x)$

Multiplying out the parentheses

$50 - 5x - 10x + x^2 = 52 - 4x - 13x + x^2 \quad | -x^2$

and combining:

$50 - 15x = 52 - 17x \quad | + 17x - 50$

$2x = 2$

$x = 1$

Check: $LS = (5 - 1)(10 - 1) = 4 \cdot 9 = 36$

$RS = (4 - 1)(13 - 1) = 3 \cdot 12 = 36$

$LS = RS$

Here, again, is opportunity to practice multiplying out parentheses. x_2 goes away, otherwise the equation would not be solvable for us.

It is also important to practice equations in which the unknown x is expressed by undetermined numbers a, b,…

13. Example:

(a + b ≠ 0)

$$\frac{x}{a} + \frac{x}{b} = 1 \ | \cdot ab \ \rightarrow \ bx + ax = ab \ \rightarrow \ (b + a)\ x = ab \ \rightarrow \ x = \frac{ab}{a + b}$$

Check:

$$LS = \frac{\frac{ab}{a+b}}{a} + \frac{\frac{ab}{a+b}}{b} = \frac{ab}{a \cdot (a+b)} + \frac{ab}{b \cdot (a+b)} = \frac{b}{a+b} + \frac{a}{a+b} = \frac{a+b}{a+b} = 1 = RS \,.$$

14. Example:[57] (a + b ≠ 0)

$$\frac{a}{x} + \frac{b}{x} = 1 \,|\cdot x \ \rightarrow \ a + b = x \ \ oder \ \ x = a + b$$

Check: $LS = \frac{a}{a+b} + \frac{b}{a+b} = \frac{a+b}{a+b} = 1 = RS$

15. Example:

$$2a + a(x - a) = 1 \ \rightarrow \ a(x - a) = 1 - 2a \ \rightarrow \ x - a = \frac{1-2a}{a} \ \rightarrow$$

$$x = \frac{1-2a}{a} + a = \frac{1-2a+a^2}{a} = \frac{(1-a)^2}{a}$$

Check:

$$LS = 2a + a(x - a) = 2a + a\left[\frac{(1-a)^2}{a} - a\right] = 2a + (1-a)^2 - a^2 = 2a + 1 - 2a + a^2 - a^2 = 1 = RS$$

16. Example:

The following example requires considerably more skill. The answer check especially requires certainty of rules regarding calculating fractions and use of the binomial formulas. Whoever can successfully work through this problem has proven they are quite skillful!

$$(b - a)\,(x + a) - (a + b)\,(b - x) = 0$$

---

[57] In problem 13, up to the factor 2, it has to do with the harmonic average, and in problem 14, up to the factor ½, the arithmetic average.

Multiply out:

$bx + ab - ax - a_2 - (ab - ax + b_2 - bx) = 0$

Get rid of the parentheses:

$bx - ax + ab - a^2 - ab + ax - b^2 + bx = 0$

$2bx - a^2 - b^2 = 0 \quad | +a^2 + b^2$

$2bx = a^2 + b^2 \quad | : 2b$

$$x = \frac{a^2 + b^2}{2b}$$

Check:

$$LS = (b-a)\cdot\left(\frac{a^2+b^2}{2b}+a\right)-(a+b)\cdot\left(b-\frac{a^2+b^2}{2b}\right)=(b-a)\cdot\frac{a^2+b^2+2ab}{2b}-(a+b)\cdot\frac{2b^2-a^2-b^2}{2b}$$

$$=(b-a)\cdot\frac{(a+b)^2}{2b}-(a+b)\cdot\frac{b^2-a^2}{2b}$$

$$=\frac{(b-a)(a+b)(a+b)-(a+b)(b-a)(b+a)}{2b}=0=RS$$

One can see from the examples what steps are necessary to solve an equation:

1. Step: Take away the denominators (denominators away!)
2. Step: Get rid of the parentheses (parentheses away!)
3. Step: Put the terms in order: Terms with the unknowns on the left, the others on the right (order!)
4. Step: Combine terms on both sides (combine!)
5. Step: Isolate the unknowns (isolate!)
6. Step: Check

Naturally, in some cases when certain steps are skipped or the order of operations is changed, it can lead to a faster solution. Skill in commuting equations can only be gained through practice!

Now, we will take a look at distinctions between possible conditions, as promised: As examples, we have the following three conditions:

1. $3 \cdot (x - 1) = 2x - (2 - x) - 1$
2. $3 \cdot (x - 1) = 2x - (2 - x) + 1$
3. $3 \cdot (x - 1) = 2x - (2 - 2x) - 2$

When we try to solve the 1st condition, through commutation we get:

$3x - 3 = 3x - 3$

This means that the 1st condition for *all* numbers that will replace x is fulfilled. It has to do with an *identity*. Commuting the 2nd condition leads to this:

$3x - 3 = 3x - 1$ or $-3 = -1$

This is obviously an *inconsistency*. **(Trans. Note: German word Widerspruch can also mean contradiction. This may be decides by the American colleagues)** The second condition cannot be fulfilled by any number.

The 3rd condition leads to this:

$3x - 3 = 4x - 4 \rightarrow 1 = x \text{ or } x = 1$

This is the solution that fulfills the original equation. Therefore, we are dealing with a *solvable condition..*

Practice 52

Calculate the number that will replace x and solve the equation. Show each step. Always check your answer using the original equation.

1. a) $3x + 8 = 41$; b) $27 - 5x = -8$
2. a) $8 + \frac{x}{10} = 10$; b) $\frac{3}{4} - \frac{x}{3} = \frac{5}{12}$
3. a) $\frac{3}{2}x - \frac{7}{4} = 2$; b) $3x + \frac{1}{5} = -1$
4. a) $6x - 3 = -1$; b) $-5x + \frac{2}{3} = 4$
5. $16x - 15 + 3x = 11x + 19 - x - 16$
6. $29 - 14x + 21 - 10x - 36 + 5x = 14$
7. $8x - (52 + 3x) = 6 - (10 - x)$
8. $3 \cdot (2x - 6) = 2 \cdot (5x - 1) - 11(x - 3)$
9. $2x - \frac{1}{4}x - \frac{1}{6}x - \frac{1}{8}x - \frac{2}{3}x - \frac{11}{12}x = -3$
10. $\frac{3x - 10}{4} + \frac{x + 5}{5} = 8$
11. a) $-\frac{1}{x} = 3$; b) $-\frac{2}{x} = \frac{1}{3}$
12. a) $-1 = -\frac{1}{x}$; b) $-1 = \frac{2}{x}$; c) $-3 = \frac{6}{x}$
13. a) $1 + \frac{1}{x} = 2$; b) $1 - \frac{1}{x} = 2$
14. a) $-1 + \frac{1}{x} = 2$; b) $1 - \frac{1}{x} = -2$
15. a) $2 - \frac{5 - x}{2} = 1$; b) $2 - \frac{5 - x}{2} = 0$
16. a) $2 - \frac{5 - x}{2} = \frac{1}{2}$; b) $2 - \frac{5 - x}{2} = \frac{2}{3}$
17. a) $2 - \frac{1}{2 - x} = 1$; b) $2 + \frac{1}{2 - x} = 1$
18. a) $2 - \frac{1}{2 - x} = 0$; b) $2 + \frac{1}{2 - x} = 0$
19. Decide if the following equations have to do with an identity, a solvable condition, or an inconsistency:

a) $2x - 1 = 2 \cdot (x - 1)$ b) $0{,}5 \cdot (2x + 4) = x + 2$ c) $7x - 5 = 5x - 7$.

Solutions:

1a) *11;* b) *7*; 2a) *20;* b) *1*; 3a) $\frac{5}{2}$; b) $-\frac{2}{5}$; 4a) $\frac{1}{3}$; b) $-\frac{2}{3}$; 5. *2*; 6. *0*; 7. *12*; 8. *7*; 9. *24*; 10. *10*; 11a) $-\frac{1}{3}$; b) *–6;* 12a) *1*; b) *–2*; c) *-2*; 13a) *1*; b) *–1*; 14a) $\frac{1}{3}$; b) $\frac{1}{3}$; 15a) *3*; b) *1;* 16a) *2*; b) $\frac{7}{3}$; 17a) *1*; b) *3*; 18a) $\frac{3}{2}$; b) $\frac{5}{2}$; 19a) Inconsistency; b) Identity;

c) Solvable Condition: x = -1.

II. Word Problems

We began this chapter with a practice exercise intended to help students get insight into the study of equations. Then we practiced techniques for commuting and solving equations. Once some confidence has been gained in this area, one should return to practical usage. It gives a connection to reality and helps one to react correctly to real problems. Granted, in the history of mathematics there are many examples to be found that have little connection to practicality. They may be stimulating for intellectual training, but they do not impart any deeper understanding of the world. The teacher must decide what he or she will emphasize. I always tried to choose examples that gave valuable insights into questions of mathematics, or dealt with life skills.

Fundamentally, it matters less how many problems are solved than it does how thoroughly thought-out and understood they are. Of course, one will be able to differentiate in the classes. Some students will wish to be challenged with the technical part of equations while others will prefer problem solving. The problem solvers are often not the fastest calculators. One can experience many surprises in this regard.

In the following I will go through a series of examples that are not always simple in order to show how one can go about solving word problems. Besides pure mathematics, word problems should also be given as homework to allow the students enough time to work through them. Not everyone will be able to do them successfully. During class the other students' thoughts about solutions will be gathered and together a solution worked out that should involve everyone.

Sometimes an objection is raised that the suggested situations aren't necessarily conducive to calculating, but rather taking action. It does, however, make a lot of sense to go ahead and calculate such situations. Being able to see the applicable rules and principles helps build the capability of sound estimating and often makes possible correct, instinctive performance.

Practice 53

1. Is there an isosceles triangle which has a base angle that is double (half, third) the size of the angle at the apex? If yes, calculate all the angles.

Solution:

In every triangle, the sum of the interior angles is $\alpha + \beta + \gamma = 180^\circ$.

In an isosceles triangle the base angles α and β are the same: $\alpha = \beta$. In the first case above, the angle α should be double the size of angle γ at the apex: $\alpha = 2\gamma$. We get then:

$\alpha + \beta + \gamma = 2\gamma + 2\gamma + \gamma = 5\gamma = 180^\circ$. From the last equation we get:

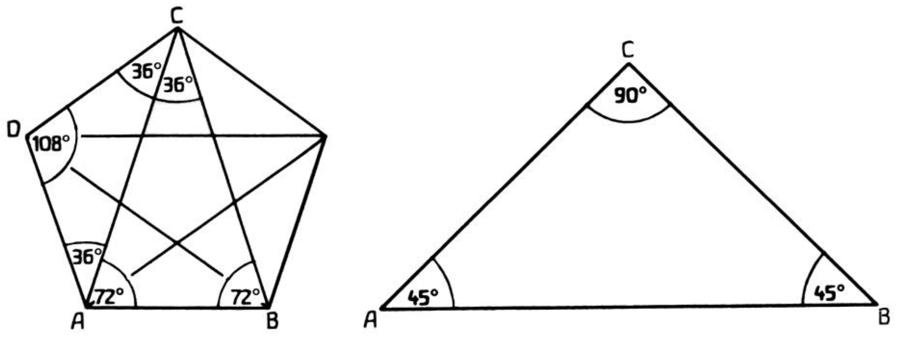
$\gamma = 36$. *So,* $\alpha = \beta = 72$. This is the triangle that appears in a pentagram.

In the second case above $\alpha = \frac{\gamma}{2}$. We then get: $\frac{\gamma}{2} + \frac{\gamma}{2} + \gamma = 2\gamma = 180^\circ$

From this we get: $\gamma = 90^\circ$. So, $\alpha = \beta = 45^\circ$. This is an isosceles right triangle, as it is used in a set square, for instance.

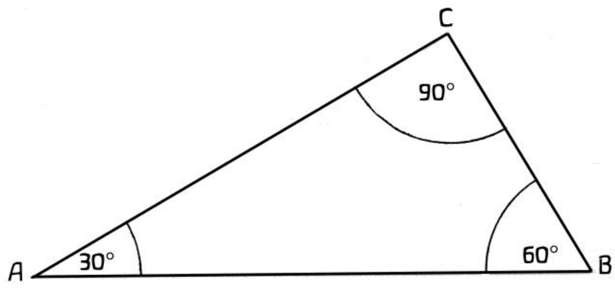
In the third case above, $\alpha = \frac{\gamma}{3}$. So, that gives: $\frac{\gamma}{3} + \frac{\gamma}{3} + \gamma = \frac{5}{3}\gamma = 180^\circ$ and $\gamma = 108^\circ$.

The base angles are then $\alpha = \beta = 36^\circ$. Those kinds of triangles appear in a regular, complete pentagon (pentagon with inscribed pentagram).



Graphic 12: Solution to the First Problem

2. Is there a triangle in which one angle is double the size of another angle, and the third angle is three times the size of the smallest? If yes, calculate all the angles.



Graphic 13: Solution to the Second Problem

Solution:

The angle should fulfill the following condition: a + 2a + 3a = 180º. From this we get: a = 30º, β = 60º, γ = 90. This is a right triangle as is often used in drawing.

3. Number a is 17 less than number b. Added together, they equal 51. What are these two numbers?

Solution:

The condition requires: a = b – 17. If we add b on both sides, we get: a + b = 2b – 17. This should equal 51: 2b – 17 = 51. Order: 2b = 68 → b = 34 → a = b -17 = 34 – 17 = 17.

4. A stimulating, comprehensive problem was conceived by Ernst Bindel.[58] "Imagine you have a number that you divide up into many fractions! When they're put together, the fractions are as large as the original number. But, if you don't have all the fractions at hand, then together they equal less than the original number. One should now divide a number so that first one-half is taken away, then one-quarter, and last, instead of taking the missing quarter, one takes only one-seventh. These three fractions together will be less than the starting number. Something will remain. With what number would 3 remain?" Bindel writes regarding this: "The problem takes on the form of a riddle. On the whole, one should see to it that this riddle problem is not handled in a short and compact way. Quite a number of fluctuating parts may be woven into the equation so that it seems fresh and lively instead of dry and square…" In light of the increasing decay in language, these lines written so long ago should really make one think. It need not contradict the brevity and clarity so rightly valued by mathematicians if it is not unjustifiably generalized. Of course, during the oral presentation, explanations are desired in every case.[59]

Solution: The unknown number is x. The condition required of x is:

$$\frac{x}{2}+\frac{x}{4}+\frac{x}{7}=x-3.$$

First, the denominators are removed by multiplying on both sides by the common denominator 28:

$$14x + 7x + 4x = 28\cdot(x - 3) \rightarrow 25x = 28x - 84$$

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[58] Ernst Bindel, *Die Arithmetik. Menschenkundliche Begruendung und paedagogische Bedeutung,* Stuttgart ^{2}1967, P. 54.

[59] G. Kowol told me about this example of a problem intended as a funny riddle: A farmer decides that each of his three sons should inherit one-half, one-third, and one-ninth of his estate. When he died there were 17 cows in the barn. The sons did not know how the cows should be rightly divided so they asked a wise man for his advice. The man recommended they borrow a cow from the neighbor and then divide them. So, the first son got 9 cows, the second 6, and the third 2, 17 all together. Now they could return the borrowed cow.

Order:

$3x = 84 \;\; \rightarrow x = 28.$

Check:

$$LS = \frac{28}{2} + \frac{28}{4} + \frac{28}{7} = 14 + 7 + 4 = 25, \quad RS = 28 - 3 = 25 \rightarrow LS = RS.$$

5. At what time between 4 and 5 o'clock are the hands of the clock exactly on top of each other; what time are they opposite each other?

Solution:

For a seventh or eighth grade class this problem requires intense effort in trying to understand the movement ratios of a clock.

First, it has to be established that the large hand goes around twelve times for every one time that the small hand revolves. The minute hand turns twelve times faster than the hour hand. We begin with the 4 o'clock position and see that the hands form an exact angle of 120º. (Why?) If the large hand turns xº, the small hand turns $\frac{x^\circ}{12}$. Both should then be in the same position. However, the smaller hand is advanced by 120º. Therefore, the condition must be:

$$x^\circ = \frac{x^\circ}{12} + 120^\circ \rightarrow \frac{11}{12} x^\circ = 120^\circ \;\; \rightarrow x^\circ = \frac{12}{11} \cdot 120^\circ .$$

This result can be further delineated into hours, minutes, and seconds: We started at 4 o'clock.

$\frac{12}{11} \cdot 120^\circ$. $\left(\frac{11}{11} + \frac{1}{11}\right) \cdot 120^\circ = 120^\circ + \frac{1}{11} \cdot 120^\circ$ 120º corresponds to 20 minutes. $\frac{1}{11}$ of 20 minutes is about 1.8 minutes. One minute is 60 seconds. 0.8 minutes is 0.8 · 60s = 48s. The clock hands are exactly over each other at 4 hrs. 21min. 48sec.

The special difficulty of this problem is that first the rotation, and then the relationship between the angles and the clock hands have to be differentiated in terms of tempo.

The opposite position or even the right-angle position can be found using a similar thought process:

For the opposite position this applies: The large hand must rotate 120º (the advance of the small hand), the rotation of the small hand $\frac{x^\circ}{12}$, plus 180º. This is the required condition:

$$x^\circ = \frac{x^\circ}{12} + 120^\circ + 180^\circ \quad \rightarrow \quad x^\circ = \frac{12}{11} \cdot 300^\circ$$

This corresponds approx. to the time: 4:54:33.

Even more interesting than clock hand positions are related problems from astronomy. However, one should not neglect to point out that the moving heavenly bodies in no way exhibit the kind of fixed ratios that are found in a clock. The moon and planets move to a wide variety of rhythms for which we are only able to calculate ratios in averages.[60]

6. The Sun gets through the zodiac, in the background of the fixed stars, one time in about 365.256 days, and the Moon takes 27.322 days (sidereal month). On average, how much time goes by between two new or full moons?

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[60] See, for example, Joachim Schultz, *Rhythmen der Sterne. Erscheinungen und Bewegungen von Sonn, Mond und Planeten*, Dornach 1985.

Solution:

Let us begin with the new moon, that is, when the Moon and the Sun have approximately the same position in the zodiac. Per day the moon travels $\frac{1}{27{,}322}\cdot 360°$, the Sun $\frac{1}{365{,}256}\cdot 360°$

After x many days the faster Moon should have traveled the complete zodiac, that is, 360º, plus the path of the Sun up to the next meeting point:

$$360° + \frac{x}{365{,}256}\cdot 360° = \frac{x}{27{,}322}\cdot 360° \quad \to \quad \frac{x}{27{,}322} - \frac{x}{365{,}256} = 1.$$

Results of the calculation:

$$x \approx 29.531 \text{ days}$$

Converted, it equals about 29 days, 12 hours, and 43 minutes. The time from one new moon (or full moon) to the next is called a *synodal month*. In actuality, the movements have various irregularities so that the calculated value can only be an average. Divergences of up to about 13 hours are possible.

7. A smelter that contains 50% copper and 50% zinc will produce a certain kind of brass if the proportions are 62% copper and 38% zinc. What amounts of the original smelt mixture and copper are necessary to produce 1000kg of brass?

Solution:

The unknown amount of smelt is x. We make the copper calculation: In x kg of the smelt there are $\frac{x}{2}kg$ of copper. There should be 620kg of copper contained in the 1000kg of brass. With x kg of smelt we have the amount to be added that is the difference between the desired 1000kg and the amount of copper already present; 1000kg – x kg pure copper. This must apply:

$$\frac{x}{2}kg + (1000 - x)kg = 620kg.$$

From the above we calculate:

$$x = 760$$

760kg of smelt and $620kg - \frac{760}{2}kg = 240kg$ copper to add in order to get the desired brass.

8. A class wants to take a class trip and investigates the transportation costs. The train will cost \$41.00 per person. A bus would cost \$950.00 for everyone. How many students must there be in order for the train to be less expensive?

Solution:

The unknown number of students x, times the train price must be smaller than the bus price:

$$x \cdot €41,- < €950,- \to x < \frac{950}{41}.$$

Since it is not possible to take a fraction of a student, we look for the largest whole number that is $\leq \frac{950}{41}$ It is the number 23. From a financial viewpoint, it would pay to take the train if there were no more than 23 students. Environmental factors have not been taken into consideration.

9. Pure gold and silver can not be used for jewelry, or most other uses, because they are too soft. In fact, the metals exhibit different properties in their highly purified form than we normally expect. For this reason, in order to make jewelry, pure gold and silver must be mixed with other metal through a smelting process to form an alloy. Of course, the cost of a piece of jewelry depends on the amount of gold or silver it contains because the other metals are significantly less expensive than gold or silver. A special look can be achieved with a gold alloy. For instance, with copper one gets a reddish tone, with silver a yellow tone, and with palladium a gray tone. In this way one can get a matching color tone to use with different precious stones or pearls. Naturally, the metals should be harder for a finger ring than for earrings or a brooch because one grasps many things with the hands. That is why people who have been married a long time often have rings that have become very thin. Soft gold would be used up much too quickly.

With gold and silver one designates the amount of pure metal in an alloy with the *fineness f*. It is measured in thousandths. So-called pure gold or silver has the fineness f = 999.9 (1000 would be the theoretic value). If f = 700, then 700 parts of every 1000 parts is pure gold or silver.

In Anglo-Saxon countries the portion of gold is given in karats. One calculates in twenty-fourths. Theoretically, pure gold consists of 24 twenty-fourths. Gold has 24 karats. If only half was gold, it would be 12 karats: 12: 24 = 1: 2.

Goldsmiths and silversmiths make the alloys in small smelting ovens, or Bunsen burners, according to what it will be used for and the price the customer is willing to pay. They start with alloys that are on hand and then smelt the desired mixture. They use small ceramic or graphite bowls that can withstand the high temperatures without burning or melting.

Example for calculating an alloy: How much silver with a fineness of $f_1 = 700$ and how much with a fineness of $f_2 = 900$ must be used in order to get 10kg of silver with a fineness of f = 780?

Solution: If we have x kg of the fineness f_1, then we have (10 – x) kg of the other fineness. In x kg of f_1 there is this much silver:

$$\frac{f_1}{1000}\cdot x\,kg = \frac{700}{1000}\cdot x\,kg = 0,7\cdot x\,kg$$

Correspondingly, in (10 – x) kg there is this amount of silver:

$$\frac{f_2}{1000}\cdot (10-x)\,kg = \frac{900}{1000}\cdot(10-x)\,kg = 0,9\cdot(10-x)\,kg$$

The 10 kg of alloy with a fineness f should contain this amount of silver:

$$\frac{f}{1000}\cdot 10\,kg = \frac{780}{1000}\cdot 10\,kg = 7,8\,kg$$

This is the calculation to determine the amount of silver: The amounts of silver contained in both alloys will determine the amount of silver in the new alloy. This must apply:

$$0,7\cdot x\,kg + 0,9\cdot(10-x)\,kg = 7,8\,kg \quad (*)$$

From this we can calculate the unknown x. First, we multiply out the parentheses and combine the x terms:

$$0,7\cdot x\,kg + 0,9\cdot(10 - x)kg = 0,7\cdot x\,kg + 9\,kg - 0,9x\,kg = 9kg - 0,2x\,kg = 7,8kg$$

When ordered, we get:

1.2 kg = 0.2x kg

By dividing with 0.2 we get:

x = 6 kg

We take 6 kg with the fineness 700 and (10 – 6) kg = 4 kg with the fineness 900 in order to get 10kg with the fineness f = 780.

By inserting the result into the equation (*), a simple probability test shows that it can be correct: 1. we have the desired 10 kg. 2. Since the fine content of f = 780 is closer to f_1 = 700 than f_2 = 900, so more of the first sort must be used.

These kinds of mixed calculations can be done similarly for many different situations. In order to make the applicable principles transparent, we will go through the following considerations:

There are two alloys with the fine content of f_1 and f_2. What needs to be produced is G kg of an alloy with a fine content of f.

Solution:

If we designate x as the needed amount with the fine content f_1, then G – x is the amount with the fine content f_2. The calculation for the amount of silver leads to:

$$\frac{f_1}{1000} \cdot x + \frac{f_2}{1000} \cdot (G - x) = \frac{f}{1000} \cdot G$$

Multiplying by 1000 gives:

$$f_1 \cdot x + f_2 (G - x) = f \cdot G.$$

From the above we get:

$$x \cdot (f_1 - f_2) = G \cdot (f - f_2) \text{ or}$$

$$x = G \cdot \frac{f_2 - f}{f_2 - f_1} \quad (1)$$

This is the needed amount of alloy with the fine content f_1. From the other alloy we need the amount G – x in order to get the total amount G. So, for the amount of the second sort we get:

$$G - x = G \cdot \frac{f - f_1}{f_2 - f_1}$$

The principle can be better understood if we calculate the *ratio* of the two alloys. It is:

$$x : (G - x) = \frac{f_2 - f}{f_2 - f_1} : \frac{f - f_1}{f_2 - f_1} = (f_2 - f) : (f - f_1)$$

(The G was not written on the right side because it could be immediately removed.)

If we name y as the amount G – x, we get:

$$x : y = (f_2 - f) : (f - f_1) \quad (2)$$

Now we can clearly see the basic underlying principle:

The needed amounts of both alloys behave just like the differences of their actual fine content to the desired fine content.

Solving this equation to find f, we get:

$$f = \frac{x \cdot f_1 + y \cdot f_2}{x + y} \quad (3)$$

Knowing that the amount of silver in the two alloys is x · f_1 or y · f_2, respectively, the result becomes immediately apparent:

The fine content of the mixture is the ratio of the silver content in each individual alloy to the total amount.

Formula (2) would result in a *negative* value if f was not positioned *between* f_1 and f_2. In our case, this would not make sense. This is clear, since the fine content of the mixture must lie *between* the fine content of each alloy.

We can now check formulas (1) and (3) using the example calculated above: f_1 = 700, f_2 = 900 and f = 780. The total amount of G was 10kg. If we insert these numbers into formula (1) we get:

$$x = G \cdot \frac{f_2 - f}{f_2 - f_1} = 10kg \cdot \frac{900 - 780}{900 - 700} = 10kg \cdot \frac{120}{200} = 6kg\,.$$

y, which is the amount added to make 10kg, is 4 kg, calculated like the above equation. Formula (3) confirms the accuracy of the calculated amounts:

$$x = G \cdot \frac{f_2 - f}{f_2 - f_1} = 10kg \cdot \frac{900 - 780}{900 - 700} = 10kg \cdot \frac{120}{200} = 6kg\,.$$

This is the desired fine content of the new alloy.

10. The knowledge gained from the previous example is generally applicable to more than just the production of metal alloys. For instance, in large coffee roasting facilities different kinds of coffee beans are mixed in order to achieve a certain taste, but also to be able to put a certain price on the product. Let us assume that the price of a certain kind of coffee beans is $1.10/kg, and the price of another kind is $1.30/kg. What is the correct mixture ratio in order to get a price of $1.25/kg?

Solution:

One can approach the problem in the same way as the previous example. We look for the appropriate formula. We take formula (2) from example 9: The mixture ratio must correspond to the ratio of the difference to the desired mixture price. Instead of using f for fine content, we will insert p for price:

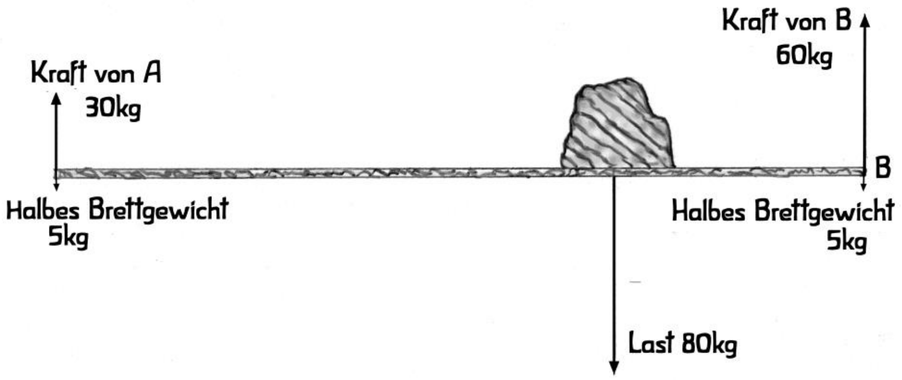
$$x : y = (P_2 - P) : (P - P_1) = (1{,}30 - 1{,}25) : (1{,}25 - 1{,}10) = 0{,}05 : 0{,}15 = 5 : 15 = 1 : 3.$$

The cheaper mixture should be mixed with the more expensive mixture in a ratio of 1:3. This appears plausible since the desired mixture price is three times closer to the higher price than to the lower price.

11. Two people who have different levels of strength want to carry a heavy load on a board. Where should they place the load so that the weight distribution corresponds to their strength levels?

Solution:

If the people each grab an end of the board, each one must carry half the weight of the board. But, we will ignore that for the moment and turn our attention to the heavy load.



Strength of A *Strength of B*

30kg *60kg*

Half of the board's weight *Half of the board's weight*

5kg *5kg*

Load 80kg

Graphic 14: For Problem 11

Let us consider the simplest cases: If the load is positioned exactly in the middle then each person carries an equal amount of weight. If the load is at the end of the board, then that person has the entire weight. The closer the load gets to one end, the heavier it is on that end.

According to the lever principle, the two loads at each end, L_1 and L_2, act in reverse to the load arms, l_1 l_2. This is the distance from each end to the load.

$$L_1 ; L_2 = l_2 : l_1$$

If, then, the loads should correspond to the ratio of their strength, they must choose the distances from the load so that they are the inverse of the strength capacities. The weaker person gets the greater distance, and the stronger person gets the shorter distance. Half of the weight of the board is added.

Example with numbers:

A 3.2 meter long board weighs 10kg and the load weighs 80kg. Person A can carry 30kg. Where must the load be placed and how much weight must the stronger person B carry?

Solution:

Since person A must carry 5kg of the board weight, they can still carry 25kg. Person B carries 55kg of the load. According to the lever principle the ratio of the length is the reverse of the ratio of the loads:

$$l_2 : l_1 = L_1 : L_2 = 25kg : 55kg = 5 : 11.$$

If we divide the board into 5 + 11 = 16 equal parts, then each part is 0.2meters. The load should be positioned so that the distance l_1 to A is 11 · 0.2m = 2.2m. Then l_2 is 1.0m, and B carries 55kg + 5kg = 60kg.

Graphic 15: For Problem 12

12. A man is not able to carry a 150kg oak beam by himself. He asks his wife for help. But half of the weight (75kg) is still too heavy for her. How will they be able to carry the beam if the woman is able to carry only 50kg?

Solution:

The weight that each person has to carry depends upon *where* they grasp the beam. If the man is at the middle of the beam then he carries all the weight. The more he goes toward the end of the beam, the heavier it will be for the other person.

If person B leans to the middle, one can imagine that the beam is divided into segments. Let us imagine for a moment that someone is supporting the beam at C. C is positioned the same distance behind B as B is from the end of the beam. We will call this length x. The length AC will be called y. B carries the two segments x so that they remain in balance. A does not carry them at all. Person A must carry half the weight of y. This is how we get closer to the solution:

The total length is l = 2x + y = 6m. The total weight of the beam is G = 150kg. The weight of one segment corresponds to its length. Besides that, we know that the loads L_A and L_B, that both people have to carry, together must correspond to the total weight G: $L_A + L_B = G$. From this we can create a proportion:

$$\frac{y}{2} : (2x + \frac{y}{2}) = L_B : L_A.$$

This can be converted by combining what is in the parentheses:

$$\frac{y}{2} : \frac{4x + y}{2} = y : (4x + y) = L_B : L_A. \ (4)$$

We know that y = l – 2x, so we get:

$(l – 2x):[4x + (l – 2x)] = (l - 2x) : (l + 2x) = L_A : L_B.$

This means:

The load for both people acts the same as the difference between the sum of the total length and double the overlapping length.

Now, we will insert all the known amounts: The woman is able to carry 50kg. So, L_A = 50kg. The man carries the rest. So, L_B = 100kg. The length is l = 6m. All of this gives us:

(6m – 2x) : (6m + 2x) = 50kg : 100kg = 1 : 2

We must now do the equation to find x. By doing *inner term · inner term = extreme term · extreme term*, we get:

6m + 2x = 2 · (6m – 2x)

It follows that:

6m + 2x = 12m – 4x

Converting leads to:

6x = 6m and x = 1m

In order to arrive at the desired division of the load the man must overlap the end by 1 meter or, to put it another way, he must move to the inside by $\frac{1}{6}$ of the total length. So, two people of different strength capacities can still carry something that neither one could do alone. This is the best way to go about this task in practice: First, both people put one end of the beam on a raised support that is far enough toward the middle. Then, the man helps his wife to put the other end on her shoulder. Finally, the man picks up the other end far enough towards the middle that he can carry the load.

II.1 Steps to a Complete Solution

The previous examples show what steps should be taken to find the complete solution to a word problem:

Step 1: What is asked, what do we know, and what can be expected as a solution? Then, one must decide what amount should be used for the variable x.

Step 2: After choosing the variable x, one must put the conditions spelled out in the problem into an equation. This is normally the most difficult step. One must always think through the original question very carefully and exactly and get a clear understanding of the meaning of the terms used.

Step 3: Once the equation for x is done, it will be solved. This requires sure confidence in the application of algebraic principles. We have already named the individual steps on page 203.

Step 4: A plausibility check can give one assurance that the size of the solution makes sense. In the actual check it will be determined if the solution really does fulfill the conditions of the equation.

Step 5: A review of the considerations and calculations is often helpful. What was the fundamental thought behind the solution? What mathematical supports did we require? Is the result what we expected?

Step 6: Whenever it appears to make sense to do so, the solution should be discussed: What did we achieve? Are there further questions to be added? Is the question too general? Where are there possible applications of the result to related cases?[61]

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[61] In his widely read book, *Arithmetik und Algebra,* 29th edition, Leipzig 1919, H. B. Luebsen writes: "Algebraic problems are so varied that they cannot be classified like specialized arithmetic: Not alone with the known, but also with the yet unknown amounts, for which one substitutes a symbol (usually one of the last letters of the alphabet, such as x, y, z, t, etc.) must arithmetic operations be undertaken that can only be indicated because there are still unknown variables.

It is not possible to have general rules for solving these problems. One must calmly consider each individual problem in order to grasp the sense of it, draw correct conclusions about the challenges and conditions expressed by words, translate that into the language of algebraic symbols, try to clothe it in an equation; namely, to put together the known and unknown amounts in such a way that the conditions of the problem according to the one is equal to the other.

After one has found the equality between the known and unknown amounts, then it is easy to find the unknown amount using the rules of the previous chapter. But, when it comes to finding the equation, then it is a matter of pure reason and acumen. Both of these things cannot be learned, but through diligent practice they can be developed.

Here, as in all mathematical applications, the mathematician is always left to his own devices. He must be certain of the conciseness of his own conclusions and differentiate between what is true and what is false. For beginners, who are not used to applying their own powers, thinking for themselves, and discovering for themselves, this creates much difficulty. But, one must not immediately shy away from it. Practice and persistent effort soon result in

Practice 54

1. A mother, living in simple circumstances, needs water at a temperature of 37º C to bathe her child. She can boil (100º C) 5 liters of water in a pot on the fire. How much water from the well that is 7º C must she add to the bath water to make it 37º C?

Solution: The unknown amount of cold water is designated as x:

$$5 \cdot 100 + x \cdot 7 = (5 + x) \cdot 37 \rightarrow 500 + 7x = 185 + 37x \rightarrow 30x = 315 \rightarrow x = 10{,}5$$

About one third boiling water and two thirds cold water will provide the desired temperature. (At home, measure the temperature of cold tap water with a bath thermometer.)

2. At 12:38 pm a train departs Munich in the direction of Augsburg. Its average speed is 130 km/h. A freight train going to Munich departs Augsburg at 12:48 pm. Its average speed is 70 km/h. When and where will the trains meet? The distance between Augsburg and Munich is approx. 60 km.

Solution:

Before the freight train departs the passenger train has already been traveling for $10min = \frac{1}{6}h$ During that time the passenger train travels 21.7km:

s = v · t

(Equation at top of page 221)

In dieser Zeit legt er nach der Beziehung: Der in der Zeit *t* zurückgelegte Weg *s* ist Geschwindigkeit *v* × Zeit *t*:

$$s = v \times t.$$

Daraus ergibt sich in den ersten 10 Minuten für den Weg des ICE

$$s = 130\frac{km}{h} \times \frac{1}{6}h \approx 21{,}7km.$$

The rest of the distance is about 38.3km. From now on the trains are traveling at the same time, each at their own speed. Naturally, up to their meeting, they are traveling the same time t. The passenger train travels:

$$s_1 = 130\frac{km}{h} \times t \quad \text{and the freight train} \quad s_2 = 70\frac{km}{h} \times t.$$

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quick understanding, correct judgment, and, finally, in a certain proficiency, for which the use of one's own power of thought not only makes things easier, but soon becomes a desire and a joy. By the way, take heed that in no scientific field does the number of examples given or undigested concepts matter in the least. A single problem that is carefully thought out, a single correct conclusion is much more useful than a thousand done in the same amount of time, but with help. Here, we present some purely algebraic problems with their solutions to show how one must approach the thing. The beginner will do well to go through them twice, the second time without looking at the answers provided. Actually, the whole of mathematics is made up of problems. The so-called algebraic problems are, however, the easiest because they do not require any special knowledge; only correct judgment and reasoning. They have been created especially for beginners and it is recommended that one take some time to practice them alone. Those who are unable to solve any algebraic problems will find everything that follows difficult and will not make rapid progress in mathematics."

Both distances together are 38.3km. Therefore:

$$s_1+s_2=130\frac{km}{h}\times t+70\frac{km}{h}\times t=200\frac{km}{h}\times t=38{,}3km.$$

From this we get for t:

$$t=\frac{38{,}3}{200}h=0{,}1915h=0{,}1915\cdot 60\,min\approx 11\,min\,29\,sec\,.$$

So, we have the following calculation:

$s_1=130\frac{km}{h}\times 0{,}1915h\approx 24{,}9km$ and $s_2=70\frac{km}{h}\times 0{,}1915h\approx 13{,}4km.$

The trains meet at approximately 13.4 km from Augsburg at about 12:29:29 pm.

(We did not take into consideration that the trains did not reach their top speed right away.)

3. In one hour a boat travels 10.2 km upstream and 17.7 km downstream. What are the speeds of the river current and the boat?

Solution:

The average distance the boat travels in one hour is given as the middle value:

$$\frac{10{,}2+17{,}6}{2}=13{,}9km$$

The river current has a speed per hour of 13.9km – 10.2km = 3.7km.