I Addendum 1: Calculating the 3rd Root (Cubic Root)

I.1 Introduction

Previously (Booklet VII, p.17) we quoted Rudolf Steiner on something he said during his Discussion with Teachers for the first Waldorf School concerning cubic roots; that one should teach them. This binomial formula is the basis for calculating square roots:

$$(a + b)^2 = a^2 + 2ab + b^2$$

Correspondingly, the basis for calculating cubic roots is this binomial formula:

$$(a + b)^3 = (a + b) \cdot (a + b) \cdot (a + b) = a^3 + 3a^2b + 3ab^2 + b^3$$

If b is again significantly less than a, the following approximation can be put to beneficial use:

$$(a + b)^3 = a^3 + 3a^2b$$

In this case the terms $3ab^2$ and b^3 do not contribute much to the final result of $(a + b)^3$.

Example 1

$$a = 80, b = 2$$

$$(a + b)^3 = (80 + 2)^3 = 512000 + 3 \cdot 6400 \cdot 2 + 3 \cdot 80 \cdot 4 + 8$$

$$= 512000 + 38400 + 960 + 8$$

$$= 551368.$$

Both of the last two terms $3ab^2$ and b^3 contribute only about 0.18% of the result.

Example 2

$$a = 1.000, b = 1$$

$$(a + b)^3 = (1.000 + 1)^3 = 1.000.000.000 + 3 \cdot 1.000.000 \cdot 1 + 3 \cdot 1.000 \cdot 1 + 1 = 1.003.003.001$$

$3ab^2$ and b^3 contribute about 0.0003% of the result.

Now, the method should be developed using the example $\sqrt[3]{238328}$.

First, we clarify: If there is a number between 1 and 10, then its 3rd exponent value will be between 1 and 1000; between 10 and 100, the 3rd exponent value will be between 1,000 and 1,000,000, etc. Conversely, this means: For every three number places in front of the decimal point of the radicand (if there are decimal places after) there is one number place in front of the decimal in the root value. Again, just as we did when finding the square root, we will designate the number groups by starting from the decimal point, going left and right, and, this time, marking 3-digit groups. The last group on the left may be either a single or double-digit number. Numbers to the right of the decimal point will be supplemented with zeros if necessary. In the result, we mark the numbers according to the number of groups in front of the decimal point with points and separate the groups in the radicand with a comma:

$$\sqrt[3]{238'328} = \;.\;\;.\;\;,$$

To determine the first approximate value of *a*, we look for the largest whole number whose 3rd exponent value is less than or equal to 238. It is the number 6 because:

$$6^3 = 216 < 238 < 7^3 = 343.$$

The tens column of the root is 6 and it is a = 60. Now, we look for the first adjustment, b, so that

$$(a + b)^3 = (60 + b)^3$$

comes as close as possible to 238328. We estimate:

$$(a + b)^3 \approx a^3 + 3a^2b = 216000 + 3 \cdot 3600 \cdot b$$

This expression should come as close as possible to 238328, that is:

$$22328 = 216000 + 3 \cdot 3600 \cdot b$$

This is equivalent to:

$$10800\, b \approx 238328 - 216000 = 22328 \quad \rightarrow \quad b \approx 22328 : 10800 \approx 2.$$

So, the first adjustment is b = 2. The b may not be too large because there still must be a place found for the ignored term $3ab^2 + b^3$.

We organize the calculations into a pattern just like we did when calculating square roots, but this time it must be a little more comprehensive. By subtracting 216000, we have subtracted a^3 from the radicand. We must still subtract $3a^2b + 3ab^2 + b^3$. The calculation is written on the left, while, on the right, the general meaning is given.

$\sqrt[3]{238'328} = 62$	$a = 60,\ b = 2$
$-216\ 000$	$-a^3$
$22\ 3'28\ :108'000 \approx 2$	$:3a^2$
$-21\ 6\ 00 = 3 \cdot 60^2 \cdot 2$	$3a^2b$
$-7\ 20 = 3 \cdot 60 \cdot 2^2$	$3ab^2$
$-8 = 2^3$	b^3
0	

Since the difference is 0, the root has been determined exactly, and it is

$$\sqrt[3]{238'328} = 62.$$

The answer check turns out correct: $62^3 = 238328$

In order to abbreviate, if we leave out the zeros and minus signs and denote the numbers only by their positions, then the calculation would look like this:

$$\sqrt[3]{238'328} = 62$$
$$-216$$
$$223'28 \quad :108 \approx 2$$
$$216 \quad = 3 \cdot 6^2 \cdot 2$$
$$7\ 2 \quad = 3 \cdot 6 \cdot 2^2$$
$$8 \quad = 2^3$$
$$0$$

If there are three or more number groups in the radicand, then the whole process is repeated in that now the approximation $a_1 + a + b$ is chosen and the next adjustment b_1 is found.

Let us look at another example:

$\sqrt[3]{19'902'511} = 200 + 70 + 1 = 271$		
$-8\ 000\ 000$		200^3
$11\ 902\ 511$	$: 120000 \approx 70$	$3 \cdot 200^2$
$-8\ 400\ 000$		$3 \cdot 200^2 \cdot 70$
$-2\ 940\ 000$		$3 \cdot 200 \cdot 70^2$
$-343\ 000$		70^3
$219\ 511$	$: 218700 \approx 1$	$3 \cdot 270^2$
$-218\ 700$		$3 \cdot 270^2 \cdot 1$
-810		$3 \cdot 270 \cdot 1^2$
-1		1^3
0		

Or, with leaving out the zeros:

$\sqrt[3]{19'902'511} = 271$		
-8		2^3
$11\ 9'02$	$: 12 \approx 7$	$3 \cdot 2^2$
$-8\ 4$		$3 \cdot 2^2 \cdot 7$
$-2\ 9\ 4$		$3 \cdot 2 \cdot 7^2$
$-3\ 43$		7^3
$2\ 19\ 5'11$	$: 2187 \approx 1$	$3 \cdot 27^2$
$-2\ 18\ 7$		$3 \cdot 27^2 \cdot 1$
$-8\ 1$		$3 \cdot 27 \cdot 1^2$
-1		1^3
0		

Therefore when finding the cubic root, (according to Mocnik) the following process should be used:

1. Starting at the decimal point, one divides the number to the left and right into number groups of 3 digits each. The last group on the left also may be a single or double-digit number. The number of groups in front of the decimal point tells us the number of digits in front of the decimal point in the result. The amount of number places is marked in the result in front of the decimal point. Then one looks for the largest number whose 3rd exponent value is contained in the first number group on the left, writes this in the result as the first digit, and subtracts its 3rd exponent value from the first number group.

2. The following digits in the root are found by division. To the current remainder, one brings down the next number group, and looks at the so-created number, for the time being without considering the last two digits, as the dividend that will be divided by the triple square of the existing result for the root. The single-digit quotient is written as a new digit in the result.

3. The portions created through the newly gained adjustment in the 3rd power, namely, $3a^2b$, $3ab^2$, and b^3 are subtracted. In the shortened written form, special attention must be

given to the correct position of the digits. The digits in the previous remained that were left unconsidered are to be included in the subtraction. If the sum of the adjustment terms can not be subtracted, then the new digit, b, is too large. It must be reduced until one is able to subtract it.

4. This process is continued. If the result does not come out even, then three zeros can be added to the radicands each time until the desired exactness is reached.

I.2 Calculating the nth Root

In principle, this method can be used to calculate any roots; even though it becomes very unwieldy. The binomial formula, or, rather, the formula for approximating in cases where *b* is very much smaller than *a*, plays the most important role.

$$(a+b)^n = \binom{n}{0}\cdot a^n + \binom{n}{1}\cdot a^{n-1}b^1 + \binom{n}{2}\cdot a^{n-2}b^2 + \ldots + \binom{n}{n-1}\cdot a^1 b^{n-1} + \binom{n}{n} b^n$$

$$(a+b)^n \approx a^n + n \cdot a^{n-1} \cdot b \quad \rightarrow \quad b \approx \frac{(a+b)^n - a^n}{n \cdot a^{n-1}}$$

If programming of algorithms is studied in the upper grades, nice examples can be taken from this.

Practice 62

1. What is the edge length of a cube whose volume is $1m^3$, $2m^3$, $3m^3$…, $8m^3$? Calculate the edge length up to two decimal places.

2. By what factor *k* must the edge length of a cube be multiplied so that the volume is doubled? Calculate up to two decimal places.

3. (This and the following problems should be done only if the formula for cube volume has been taught: $V = 4/3\pi\, r^3$.) When introducing fractions in the fourth grade, the teacher has each student make a different number of clay balls from a single clay ball that is 10cm in diameter. They all should be the same size. Some students make only two balls, some three, etc. What is the diameter of the clay balls made by the students (up to 5 balls)? Should she advise one student to make all equal balls with 1cm diameter?

4. A lead ball that is 3cm in diameter is cast into a cube form. What is the side length, *k*, of the cube?

Solutions:

1. 1m; 1.26m; 1.44m; 1.59m; 1.71m; 1.82m; 2m.

2. $k = \sqrt[3]{2} = 1.2599 = 1.26$.

3. The diameter, *d*, of the original clay ball is to the diameters of the smaller balls, d_2 (two balls), d_3 (three balls), and generally, d_n, as the cubic root of the ratio of the volumes of the balls: $d : d_n = \sqrt[3]{V : V_n} = \sqrt[3]{n}$, that is, as 1.26 : 1 (two balls), 1.44 : 1 (three balls), etc. One gets the ball diameter of

$$d_n = \frac{d}{\sqrt[3]{n}}.$$

That is, $d_2 = 7.94; d_3 = 6.93; d_4 = 6.30; d_5 = 5.85.$

From the large clay ball that is 10cm in diameter, we have: 10^3 = 1,000 balls that are 1cm in diameter!

4. *$V = 14.14cm^3$; $k = 2.418cm$.*

II Addendum 2: The Euclidean Algorithm

II.1 Introduction

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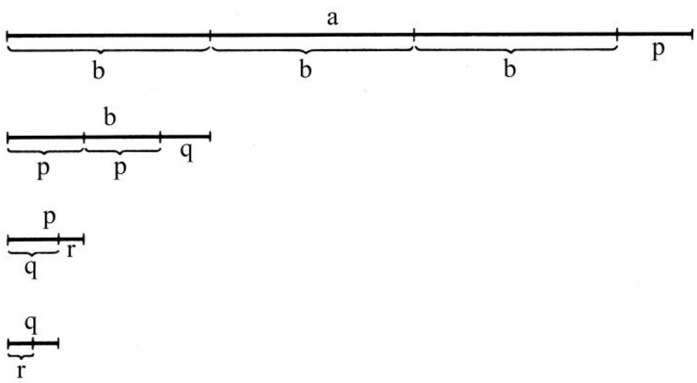
It can not be assumed that a length used for measuring will come out evenly in every length that is measured. This applies, for instance, when one measures a 12m length using a 5m length. First, one looks for a common measure for both lengths. Ancient Greek builders developed a very practical method: Let us say there is an existing square structure with sides *a* and *b* and one wants to put some ornamental decoration on the structure so that it fits evenly on both sides. First, one takes a string that is the same size as the shorter length, *b,* and measures it against a string that is the same size as the longer length, *a*. If there is no remainder, then each regular division of the shorter side also will fit into the longer side with no remainder (1st Case). If there is a remainder, r_1, then one checks to see if r_1 will fit evenly into the shorter side, *b* (2nd Case). It that was the case, then every whole-number portion of r_1 must fit evenly into *b* and *a*. If r_1 does *not* fit evenly into *b*, there is a remainder, r_2. If r_2 fits evenly into r_1, then every whole-number portion of *r2* must also fit evenly into *b* and *a* (3rd Case). If that was not the case, then one continues in this way until a remainder fits evenly, or is so small that one does not need to worry about it. This process is called the *Euclidean Algorithm*.[73]



Graphic 16: The Euclidean Algorithm

Even in those days, builders had to accept the limitations of precision that their material allowed. Therefore, in this way, for practical use, one can always find a common measure that is exact enough in just a few steps.

What works with measured lengths is still easier with numbers: As an example, let us ask about the greatest common **factor (Trans. Note: German word Teiler could also mean divisor)** of two numbers, because, when dealing with numbers that is the equivalent of finding the largest common measure.

[73] Euclid lived from approx. 325 to 265 B.C. He taught in Alexandria in Egypt. A mathematical operation in which the result is calculated by continually repeating the same steps is called an algorithm.

1. *Example:* Determine the greatest common factor of 120 and 72.

Step 1: We subtract the smaller number from the larger number as many times as possible, and determine the remainder:

$$120 = 1 \cdot 72 + 48$$

Step 2: Since there is a remainder of 48, we will subtract it from the smaller number (72) as many times as possible and determine the remainder:

$$72 = 1 \cdot 48 + 24$$

Step 3: Since there is a remainder still, we repeat the process with the last remainder (24) and the quantity (48):

$$48 = 2 \cdot 24 + 0$$

Now the remainder is 0, so 24 is the greatest common factor of 120 and 72. It is:

$$120 = 5 \cdot 24 \text{ and } 72 = 3 \cdot 24$$

A greater common factor can not be found.[74]

2. *Example:* Determine the greatest common factor of 123 and 81.

Solution: Use the same steps as in the previous example

$$123 = 1 \cdot 81 + 42$$
$$81 = 1 \cdot 42 + 39$$
$$42 = 1 \cdot 39 + 3$$
$$39 = 13 \cdot 3 + 0$$

The greatest common factor of 123 and 81 is 3. $123 = 41 \cdot 3$ and $81 = 27 \cdot 3$.

3. *Example:* Determine the greatest common factor of 336 and 91.

$$336 = 3 \cdot 91 + 63$$
$$91 = 1 \cdot 63 + 28$$
$$63 = 2 \cdot 28 + 7$$
$$28 = 4 \cdot 7 + 0$$

The greatest common factor of 336 and 91 is 7. $336 = 48 \cdot 7$ and $91 = 13 \cdot 7$.

4. *Example:* Determine the greatest common factor of 969 and 627.

$$969 = 1 \cdot 627 + 342$$
$$627 = 1 \cdot 342 + 285$$
$$342 = 1 \cdot 285 + 57$$
$$285 = 5 \cdot 57 + 0$$

The greatest common factor of 969 and 627 is 57. $969 = 17 \cdot 57$ and $627 = 11 \cdot 57$.

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[74] Here, it would not be difficult to prove that the Euclidean Algorithm really does determine the largest common factor, but it is not yet the appropriate place to do so. See Louis Locher –Ernst, *Arithmetik und Algebra,* Page 260 ff.

This applies to all of the examples: If the quantity numbers used represent measured lengths – regardless of the unit of measure – then the greatest common factor is a length that will fit evenly into both lengths. If one thinks of two liquid measurements – regardless of the unit of measure – then both amounts would be completely utilized with one quantity that corresponded to the greatest common factor. If one is working geometrically, this process tells the greatest common unit (length, area, volume, etc.) to the comparison of both starting measurements.

Through measuring, we carry numbers into geometry, then, by checking how many times one line segment is contained in another, we count and form the measured value. All measuring originates from a process of repetition. Like an archetype of measuring, we carry the various rhythms within us. Over a longer time period, our breathing and heart rhythm has an average ratio of 1 : 4.[75]

Practice 63

1. Use the flexible process we have described as the Euclidean Algorithm on the following pairs of line segments:

a) **AB = 68cm,* **CD = 16cm,*

b) **AB = 75m,* **CD = 35m,*

c) **AB = 3,5m,* **CD = 0,31m,*

d) **AB = 144cm,* **CD = 81cm*

2. Take an A-4 size sheet of paper and, with the help of a compass, try to find a common measure for both sides. The attempt at a solution proves to be difficult. There are always remainders.

Solutions:

1 a) 68 = 4 · 16 + 4; 16 = 4 · 4 + 0; 4cm is the greatest common measure of both line segments.

b) 75 = 2 · 35 + 5; 35 = 7 · 5 + 0; 5cm is the greatest common measure.

c) 3.50 = 11 · 0.31 + 0.09; 31 = 3 · 9 + 4; 9 = 2 · 4 + 1; 4 = 4 · 1 + 0; 1cm is the greatest common

measure.

d) 144 = 1 · 81 + 63; 81 = 1 · 63 + 18; 63 = 3 · 18 + 9; 18 = 2 · 9 + 0; 9cm is the greatest common

measure.

2) This has to do with the fact that an A-4 format has a side ratio of √2 : 1.

II.2 Incommensurable Line Segments

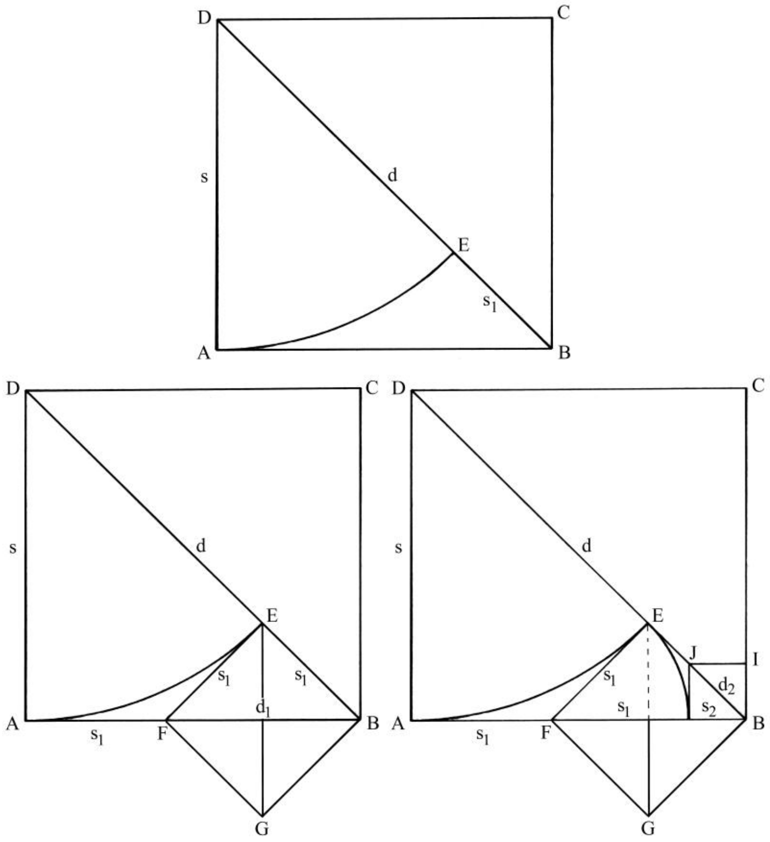
The *practical* determination of a common measure for two measurements of similar type is generally successful when done in the described way. However, if a figure is determined purely through thought – such as a geometric square or hexagon - , then there are unexpected difficulties when trying to determine a common measure. There are such things as *commensurable* and *incommensurable* line segments.

[75] See http://www.digipharm.de/07%20Chronomedizin/0712%20Chronomedizin%20Puls-Atem-Frequenz.html

We will discuss two cases of incommensurable line segment ratios in a little more detail, but without going into a very strict presentation.

II.2.1 Incommensurable Line Segments in a Square

We will start with a square ABCD and designate the diagonal BD with *d* and the sides with *s*. We will now use the Euclidean Algorithm method to find the common measure, but we will use it in a slightly different form. While doing this with the class, one should definitely not load up the drawing with letters, but rather use colors to designate specific line segments.

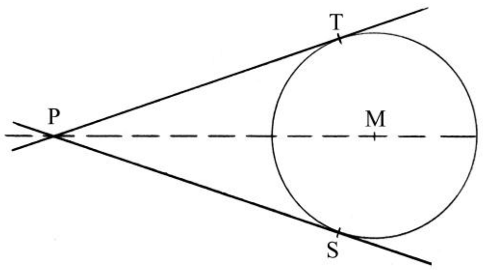


Graphic 17: Diagonals and Sides in a Square are Incommensurable

With the first step we get:

$$d - s = \overline{EB} = s_1$$

In E we construct the perpendicular and get the line segment EF. Because of the right angles and the two 45° angles in F and B, the triangle ΔFBE is isosceles and right-angled. Therefore, we can add the square FGBE. Now, FA and FE are tangent segments around the circle D with the radius *s*. FA and FE are, therefore, of equal length, as the drawing shows:



Graphic 18: The Tangent Segments are of Equal Length

PM is a symmetry axis for the whole figure. Therefore, PS = PT.

If *d* and *s* of the original square have a common measure, *m*, then *m* also must be a common measure of *s* and s_1. If we now construct $s - s_1 = d_1$, we see that *m* also must be a common measure of d_1 and s_1. However, these are diagonals and sides of the smaller square FGBE.

If we continue to apply the method by employing the same steps as in the beginning for the square FGBE, first, we get the square KBIJ, and then we get a series of increasingly smaller squares that are pushing null, whose sides are always reduced in the same ratio to the previous square. The assumed common measure of *d* and *s* (*m*), had a specific length, so there is certainly a square in the continued succession whose sides are shorter than *m*. Therefore, *m* can no longer be a common measure for the diagonals and sides of this smaller square. There can not be a common measure for the diagonals and side lengths of a square.[76]

One could make this argument even though all the steps in the proof had not yet been completed. What is especially not strictly proven is that the succession of created side lengths, with their respective diagonal lengths, will, in fact, become smaller than each previous length. For an upper class, the wonderful work of Alexander Israel Wittenberg is recommended on this subject, which is likely to remain vivid in one's memory.[77]

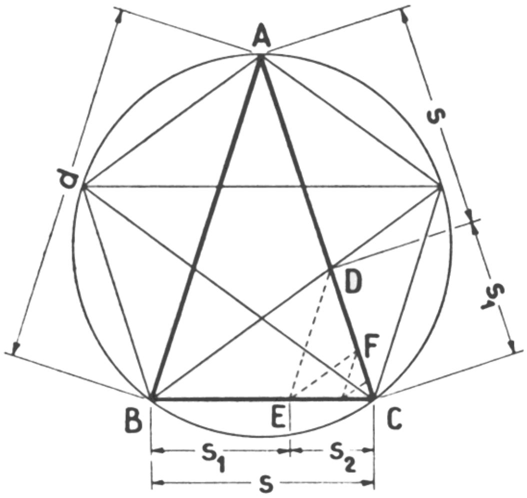
II.2.2 11.1.2 Incommensurable Line Segments in a Regular Pentagon

The diagonals a and the sides b of a regular pentagon also have no common measure. Think back to the angles in a regular pentagon as we learned about them in volume 3 of the series *Der Geometrieunterricht an Waldorfschulen.*[78]

[76] See also the chapter titled *Qualitative Betrachtungen zum Begriff des Irrationalen,* in: Gerhard Kowol, *Gleichungen. Eine historisch-phaenomenologische Darstellung,* Pages 105 – 107, Stuttgart, 1990. Also, the proof on page 99.

[77] Alexander Israel Wittenberg, *Vom Denken in Begriffen. Mathematik als Experiment des reinen Denkens,* Basel and Stuttgart, 1957.

[78] Ernst Schuberth, *Der Geometrieunterricht an Waldorfschulen. Volume 3: Die ersten Schritte in die beweisende Geometrie,* Stuttgart, 2001, Page 16. See also Louis Locher-Ernst, *Arithmetik und Algebra,* Page 188 f.



Graphic 19: Incommensurable Line Segments in a Pentagon

An isosceles triangle ΔABC has the angles 36º (A) and 72º (B and C). The triangle ΔBCD has an angle of 72º for C, and 36º for B. So, the angle for D is again 72º. ΔBCD has the same angles as ΔABC, and it is BD = BC = *b*. Since the angles A and B in ΔABC are both 36º, then also AD = BD = *s*.

We now apply the flexible subtraction method to *d* and *s*, and we get:

$d - s = s_1$

If we measure *s* with s_1, we get $s - s_1 = s_2$, whereby s_2 from s_1 and *s* comes out just like before with s_1 from *s* and *d*. So, one gets a chain of equations that is represented geometrically as ABDEF…. The line segments s_1, s_2, s_3, s_4,… become increasingly smaller. They can be less and less differentiated from 0, without breaking the chain. From this it follows that all the ratios

$s_1 : s_2, s_2 : s_3, s_3 : s_4 \ldots$

are equal, and each successive line segment is smaller than the one before, and the succession of line segments will become limitlessly small, until they get to zero. Just as with the square, we now conclude that there can be no common measure for *d* and *s*.

II.2.3 Rational and Irrational Numbers

Two line segments with a common measure are called *commensurable* line segments. If two line segments have no common measure, they are called *incommensurable*. The ratio of commensurable line segments can be expressed by a natural number or a fraction. The ratio of incommensurable line segments can be expressed by *irrational numbers*. In arithmetic, we have come across them as numbers like √2, √3, or also $\sqrt[3]{100}$, which can not be expressed by a normal fraction or a finite decimal fraction. "*Irrational*" means that they are *not measurable*. *"Ratio"* means *"reason*", but also *measure,* and the prefix "*ir"* expresses the negation of what comes after.

If we apply the Pythagorean Theorem to a square with the side *a* and diagonal *d,* we get:

$d_2 = a^2 + a^2 = 2a^2$

We calculate the root on both sides and get:

$d = a \cdot \sqrt{2}$ *or* $d : a = \sqrt{2}$

Therefore, the ratio of the diagonal to side in a square is the irrational number √2. The incommensurability of the two line segments *d* and *a* is expressed arithmetically by the fact that their ratio can not be expressed by a normal fraction or finite decimal fraction. If that was the case, one would have found a common measure for the diagonal and side of the square.