I Introduction to Higher Mathematical Operations

I.1 Introduction

This chapter seems to deal with pure formalistic things but understanding it better and better it leads to a wonderful harmony and a network of inner relations. Imagine what numbers for us were without the operations that make us the numbers connected to each other.

To make this better understandable let me tell you a very old story that is well known. Once the old Greek sage Pythagoras was asked: What is the essence of friendship? His answer was: Two people are real friends if there relation is like the relation of the numbers 220 and 284.

Can you understand this answer? It may become clearer if you look at the factors (or divisors) of the two numbers. 220 has– beside the number itself - the factors 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110.

284 has the factors 1, 2, 4, 71, 142. The *sum* of the factors of a number – except the number itself – is called the *content* of that number. We can say that the number rules its factors. They are embraced by the number. Now, when we add up the factors of 220 we get 284, and when we add up the factors of 284 we get 220! Each number equals the content of the other number. Isn't that friendship? What is not friendship? E.g. if a person says: I am such a nice person, everybody loves me, I have so many friends. Actually the person loves to be loved and mostly she loves herself. Real friends have as the content of their souls the other person. They are interested in the other. They want to know the thoughts, feelings, and intentions of them. They love to talk to each other, to do things together, and they can stand the different opinions, the different feelings, behaviors, and intentions. He as a total becomes the content of my soul.

Didn't give Pythagoras a wonderful sage answer? We can understand it if we relate numbers to each other. Factors of a number are related to it by multiplication or division; the content can be calculated if we know what a sum is, if we can add.

In the following chapter we try to explain the wonderful organism of the nine fundamental operations that make the kingdom of numbers so rich.

I.2 Inner connections among the operations

It was Rudolf Steiner, the leader of the first Waldorf school, who suggested that the higher mathematical operations be allowed to flow (develop?) from the lower operations in a certain way. As a suggestion, this is presented here in more detail, whereby we will repeat some things in order to make the correlation.[62]

I.1.1 From the Sum to the Product (From Adding to Multiplying)

First, we will look at normal addition of numbers that lead to sums in the familiar way. Here are some examples:

$5 = 3 + 2$ *or* $7 = 3 + 4$ *or* $2 = 1 + 1$

There are also sums that contain more than two summands:

$12 = 3 + 4 + 5$ *or* $20 = 1 + 2 + 3 + 4 + 4 + 4 + 3 + 2 + 1$ *or*

$18 = 4 + 3 + 4 + 3 + 4$

Among sums there are those that have *only the same* summands, like these, for example:

$$12 = \underbrace{2+2+2+2+2+2}_{6} \quad \text{or} \quad 12 = \underbrace{3+3+3+3}_{4} \quad \text{or} \quad 12 = \underbrace{4+4+4}_{3}.$$

In such cases we count the *summands* and create a *product*. In the first example we have:

$12 = 6 \cdot 2$*; in the second* $12 = 4 \cdot 3$*; and in the third* $12 = 3 \cdot 4$*.*

The counted summands that are always the same are called multiplicands; the counted number is the *multiplier.* It tells us *how often* the same summand occurs.

What has been presented by example can be generally stated as follows:

Simple, normal sums have two terms (summands):

$S_1 = a + b$

But there are also sums that have multiple terms:

$S_2 = a + b + c + \ldots$

Among the sums with multiple terms are those in which all the summands are the same. For example, if there are two equal summands as in:

$$S_3 = \underbrace{a+a}_{2},$$

We then count the summands and write:

$S_3 = 2 \cdot a$

If the sum has three terms and they are all the same, we have:

[62] See Rudolf Steiner, *Erziehungskunst. Seminarbesprechungen und Lehrplanvortraege,* Complete Works 295, 14th Seminarbesprechung. In the book *Der Anfangsunterricht in der Mathematik an Waldorfschulen,* the connection between the mathematical operations is described from a somewhat different viewpoint. For a systematic overview, it is recommended that the teacher have a clear understanding of this connection.

$$S_4 = \underbrace{a+a+a}_{3}$$

Written as a product it would be:

$$S_4 = 3 \cdot a$$

If we have an undetermined amount n of equal summands a:

$$S = \underbrace{a + a + a + a + \ldots}_{n} \ldots \textit{(n summands)}$$

Then we would write:

$$S = n \cdot a$$

We have now very generally described (in terms of comprehension) the transition from addition to multiplication. We could carry out a corresponding process – on the blackboard, if possible – by starting with the product instead of the sum. With that we come to the higher operation of *raising numbers to a given power.*

I.1.2 From the Product to the Power (From Multiplying to Raising to a Given Power)

First, we will look at normal *multiplication* of numbers that result in *products*. Here are some examples:

$6 = 3 \cdot 2$ *or* $12 = 3 \cdot 4$ *or* $2 = 2 \cdot 1$

There are products that contain more than two factors:

$12 = 2 \cdot 3 \cdot 2$ *or* $20 = 2 \cdot 5 \cdot 2$ *or* $720 = 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6$

Among products there are those that have *only the same factors*, like these, for example:

$$64 = \underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{6} \quad \text{or also} \quad 81 = \underbrace{3 \cdot 3 \cdot 3 \cdot 3}_{4} \quad \text{or} \quad 36 = \underbrace{6 \cdot 6}_{2}$$

In such cases we *count* the factors and create a *power*. The counted factors that are always the same is referred to as the *base number*. The number of factors counted is called the *exponent*. It tells us *how often* the same factor occurs. For a reason that will soon be explained, we write the base number in the normal position and the exponent is written to the right of the base number and raised. Our three examples from above would look like this:

$64 = 2^6$, $81 = 3^4$, *and* $36 = 6^2$

For the time being, and different than with multiplication and addition, here we are only dealing with *pure* numbers.[63]

There is a sensible reason as to why the base number and the exponent are not written on the same level: If we compare 2^3 and 3^2, we see that the two powers do not have the same value:

$2^3 = 8$ *but* $3^2 = 9$ *or*

$3^4 = 81$ *but* $4^3 = 64$

In general, the base number and the exponent may not be exchanged. That is why they are written on different levels.[64]

[63] It should be remembered that we have already earlier, in the third grade, used the shortened way of writing the second and third powers. (This volume is not yet published.)

[64] This form originated with R. Descartes and has achieved acceptance. He did not give a reason for choosing this form. See J. Tropfke, *Geschichte der Elementarmathematik*, 4th Edition, Volume 1, *Arithmetik und*

What has been presented by example can be generally stated as follows:

Simple, normal products have two factors:

$P_1 = a \cdot b$

But there are also products that have multiple factors:

$P_2 = a \cdot b \cdot c \ldots$

Among the products that have multiple factors there are those in which *all the factors are the same*. If, for instance, there are two equal factors:

$$P_3 = \underbrace{a \cdot a}_{2}$$

We then count the factors and write:

$P_3 = a^2$

If a product has three equal factors, we write:

$$P_4 = \underbrace{a \cdot a \cdot a}_{3}$$

Written as a power, it would look like this:

$P_4 = a^3$

If we have an undetermined number n of equal factors a:

$$P = \underbrace{a \cdot a \cdot a \cdot a \cdot \ldots \cdot a}_{n}$$

We write it like this:

$P = a^n$

We have now very generally described (in terms of comprehension) the transition from multiplying to raising to a given power.

Up to this point, an exponent has been one of the natural numbers 2, 3, 4,… Now, we will introduce: $a^1 = a$. Later, we will see that one can always look at a^0 as 1 (for $a \neq 0$). In the upper classes negative numbers and fractions will also occur as powers.

Practice 55

1. Re-create the table of first powers from page 112 and, optionally, expand it.
2. Represent the numbers 64, 128, 256, 6,561 and 10,000 powers in as many different ways as possible.
3. Look for numbers between 0 and 30 that can be either sums or differences of two square numbers. Are there also square numbers among them? Examples: $1 = 0^2 + 1^2$, $2 = 1^2 + 1^2$, $3 = 2^2 - 1^2$, $4 = 0^2 + 2^2$, $5 = 1^2 + 2^2$
4. Look for numbers between 0 and 30 that can be either sums or differences of cubic numbers. Are there also cubic numbers among them?
5. Powers of the number ten have special names. We call 10^1 ten, 10^2 hundred, 10^3 thousand, 10^6 million, 10^9, billion, 10^{12} trillion, 10^{15} quadrillion, 10^{18} quintillion, 10^{21} sextillion, etc. In order to better read the numbers one puts commas after every three digits. Read the numbers 111,222,333; 444,333,222,111; 9,876,543,210; 12,345,678,987,654,321.

Algebra (1980), Page 227. The fact that other forms have become common in not important here.

6. For a better overview, or for other reasons, one likes to represent large numbers as powers of ten. For example, 912436752 = $9 \cdot 10^8$, or, more exactly, $9.1 \cdot 10^8$. Write the numbers in the previous problem as powers of 10 with two reliably calculated digits.

Solutions:

1. Table found on page 112.

2. $64 = 64^1 = 8^2 = 4^3 = 2^6$; $128 = 2^7$; $256 = 4^4 = 2^8$; $6561 = 81^2 = 9^4 = 3^8$; $10.000 = 100^2 = 10^4$.

3. $7 = 4^2 – 3^2$; $8 = 2^2 + 2^2 = 2^3$; $9 = 0^2 + 3^2 = 5^2 – 4^2$; $10 = 1^2 + 3^2$; $11 = 6^2 – 5^2$; $13 = 2^2 + 3^2 = 7^2 – 6^2$; $15 = 8^2 – 7^2$; $16 = 0^2 + 4^2$; $17 = 1^2 + 4^2 = 9^2 – 8^2$; $18 = 3^2 + 3^2$; $19 = 10^2 – 9^2$; $20 = 2^2 + 4^2 = 6^2 - 4^2$; $21 = 11^2 – 10^2$; $23 = 12^2 – 11^2$; $24 = 5^2 – 1^2$; $25 = 3^2 + 4^2 = 13^2 – 12^2$; $26 = 5^2 + 1^2$; $27 = 14^2 – 13^2 = 3^3$; $29 = 15^2 – 14^2$.

4. With only two terms (not using zero) one gets 7 = 23– 13; 9 = 32= 23+ 13; 19 = 33– 23; 26 = 33– 13; 28 = 33+ 13. If there are more than two numbers permitted there are many further possibilities, such as: : $0^3 = 6^3 – 5^3 – 4^3 – 3^3$; $3^3 = 6^3 – 5^3 – 4^3 = 5^3 – 4^3 – 3^3 – 2^3 + 1^3$, and so on.

5. (Written in short form:) 111 million 222 thousand 333; 444 billion 333 million 222 thousand 111; 9 billion 876 million 543 thousand 210; 12 quadrillion 345 trillion 678 billion 987 million 654 thousand 321.

6. $1,1 \cdot 10^8$; $4,4 \cdot 10^{11}$; $9,9 \cdot 10^9$; $1,2 \cdot 10^{16}$.

I.3 The Rules of Number Powers

From the determination that we understand the power P = a^n to be the product $\underbrace{a \cdot a \cdot a \cdot a \cdot \ldots \cdot a}_{n}$, we immediately have the first *rule of number powers:*

First rule of number powers:

$$a^n \cdot a^m = a^{n+m}$$

In words: Number powers with the same base number, a, can be multiplied by raising a to the power of the sum of the exponents.

Second rule of number powers:

$$\frac{a^n}{a^m} = a^{n-m}$$ *, whereby we assume:* $n \geq m$.

In words: Number powers with the same base number, a, can be divided by raising a to the power of the difference of the exponents (nominator exponent – denominator exponent).

Third rule of number powers:

$$a^n \cdot b^n = (a \cdot b)^n$$

In words: Number powers with the same base number can be multiplied by raising the product of the base numbers to the power of the combined exponents.

Fourth rule of number powers:

$$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$$

In words: Number powers with the same exponent n can be divided by raising the quotients of the base numbers to the power of the combined exponents, n.

Please note: The power of a sum or a difference is generally not the same as the sum or difference of the powers of single summands:

$(a + b)^n \neq a^n + b^n$ *or* $(a - b)^n \neq a^n - b^n$

Examples:

$(2 + 3)^2 = 5^2 = 25 \neq 2^2 + 3^2 = 4 + 9 = 13$ *oder*

$(5 - 3)^2 = 2^2 = 4 \neq 5^2 - 3^2 = 25 - 9 = 16$

One can draw a few important conclusions from these rules of number powers, which will be expanded upon in the upper classes:

What is a^1? If the rules of number powers are to apply without limitations, then according to the third rule one can say:

$$a^1 = a^{3-2} = \frac{a^3}{a^2} = \frac{a \cdot a \cdot a}{a \cdot a} = a,\ \ a \neq 0.$$

In words: The first power of all numbers $a \neq 0$ is a.

What is a^0? Similarly, we can say that a^0 is a^{2-2}. With that, we have:

$$a^0 = a^{2-2} = \frac{a^2}{a^2} = \frac{a \cdot a}{a \cdot a} = 1.$$

In words: The zero power of all numbers a ($a \neq 0$) is 1.

Further: All powers of 1 are 1: $1^n = 1$ for all numbers n.

Practice 56

1. Calculate*: :* $2^1; 1^2; 2^3; 3^2; 2^4; 4^2; 2^5; 5^2; 2^6; 6^2$.
2. *2 Write the numbers 64, 81, and 1024 as powers in as many different ways as possible.*
3. With what should 5^{12} be multiplied in order to get 10^{12}?
4. Calculate: $\left(\frac{1}{2}\right)^4; \left(\frac{2}{3}\right)^3; \left(\frac{3}{4}\right)^2; \left(\frac{4}{5}\right)^1; \left(\frac{5}{6}\right)^0.$
5. Calculate: $\frac{8^3}{4^3}; \frac{15^2}{5^2}; \frac{12^4}{6^4}; \frac{7^3}{21^3}; \frac{9^2}{27^2}; \frac{a^5}{a^2}; \frac{a^2}{a^5}.$

Solutions:

1. 2; 1; 8; 9; 16; 16 (!); 32; 25; 64; 36.
2. $64^1 = 8^2 = 4^3 = 2^6$; $81^1 = 9^2 = 3^4$; $1024^1 = 32^2 = 4^5 = 2^{10}$.
3. 2^{12}, since $5^{12} \cdot 2^{12} = (5 \cdot 2)^{12} = 10^{12}$.
4. $\left(\frac{1}{2}\right)^4 = \frac{1}{16}; \left(\frac{2}{3}\right)^3 = \frac{8}{27}; \left(\frac{3}{4}\right)^2 = \frac{9}{16}; \left(\frac{4}{5}\right)^1 = \frac{4}{5}; \left(\frac{5}{6}\right)^0 = 1.$
5. $\frac{8^3}{4^3} = \left(\frac{8}{4}\right)^3 = 2^3 = 8;$ correspondingly: $\frac{15^2}{5^2} = 3^2 = 9; \frac{12^4}{6^4} = 2^4 = 16;$
6. $\frac{7^3}{21^3} = \left(\frac{1}{3}\right)^3 = \frac{1}{27}; \frac{9^2}{27^2} = \frac{1}{9}; \frac{a^5}{a^2} = a^3; \frac{a^2}{a^5} = \frac{1}{a^3}.$

I.4 Inversions of Power Functions

If we are given a power equation of $p^a = r$ then we can use two of the three numbers to calculate the third. If we calculate the r then we have to raise the number p to the power of a. However, if we ask: What number must be the exponent of 5 in order to get 125 then we are calculating to find the exponent a. Naturally, the answer is a = 3 since $5^3 = 125$. A calculation that is asking for the exponent when the base number and the result are given is called finding the *logarithm*. The shorter form of such a word problem is written like this:

$$log_5\ 125 = 3$$

One reads it like this: The logarithm of 125 to base 5 is equal to 3. The expression $\log_5$ 125 and, accordingly, the number 3, is called the logarithm of 125 to base 5. In general, one describes the expression $\log_p r$ as the logarithm of r to base p.

One can also calculate to find p in $p^a = r$. Such a problem would be: What number p must I raise to the power of 3 in order to get 125? The answer is p = 5 since $5^3 = 125$. The calculation that asks for a base number when the exponent and the result are given is called *finding the root* or *root extraction*. The word problem written in short form looks like this:

$$\sqrt[3]{125} = 5$$

One reads it like this: The cubic root of 125 is 5.

If one turns around the power equation $p^a = r$ and asks for the base p, then p is the a-th root of r and is written as p = a√r. Left of the root symbol one writes the number that tells us the number of the root to be found. Underneath the root symbol is the number from which the root is determined. The *root exponent* is a, and r is the *radicand*.[65] In our example 3 is the root exponent and 125 is the radicand.

Questions relating to the logarithms of both of the inversion operations are:

- What is the logarithm of r to base p?
- The whole r is given and the base number p. How many factors of p will give r?
- The p is given. The r should be an exponent of p. How often is p used as a factor?

Questions relating to the radicand:

- The whole r should be broken down into a product of a, with equal factors. How large is the single factor p?
- What amount is p, raised to the power of a, in order to get the total amount of r?

Other formulations are possible.

With the above we have become familiar with a few rules of number powers. There are also rules for root extractions and logarithms. Let us look at a rule for root extraction: What is the root of the product a · b? Let us look at a few examples: What is the square root of 16 · 25? 16 · 25 = 400 and √400 = 20. On the other hand, √16 = 4 and √25 = 5. It is, however, 20 = 4 · 5. Therefore, √16 · 25 = √16 · √25. Is it generally allowed to find the root of a product using each individual factor? Let us look at some further examples: 6 = √36 = √4 · 9; √4 · √9 = 2 · 3 = 6. Therefore, √4 · 9 = √4 · √9. Similarly, √25 · 9 = √25 · √9, as can be easily checked.

In order to really understand that $^n\sqrt{a \cdot b} = {}^n\sqrt{a} \cdot {}^n\sqrt{b}$, we must only bring the clear meaning of the root into our consciousness again: With $^n\sqrt{r}$ it is understood to be the number that is raised to the power of n to get r. Therefore, in our case, it is: $({}^n\sqrt{a \cdot b})^n = a \cdot b$. On the other hand, according to the third rule of powers: $({}^n\sqrt{a} \cdot {}^n\sqrt{b})^n = ({}^n\sqrt{a})^n \cdot ({}^n\sqrt{b})^n = a \cdot b$. Therefore, this applies:

First Rule of Roots:

$$^n\sqrt{a \cdot b} = {}^n\sqrt{a} \cdot {}^n\sqrt{b}$$

In words: *From a product, the root may be extracted from the factors; or, conversely: Two*

[65] For information about the origination of the root symbol compare J. Tropfke, *Geschichte der Elementarmathematik*, Volume I, Berlin [4]1980. See also Louis Locher-Ernst, *Arithmetik und Algebra*, Dornach [2]1984, Page 234 and following pages.

roots with the same root exponents can be multiplied together by multiplying the numbers under the roots and extracting the root from the product.

Please note: One may never extract the root of a *sum* using its individual terms. Unfortunately, this is one of the most common mistakes when dealing with roots and should be often emphasized. An example immediately shows the inequality:

$$\sqrt{9+16} = \sqrt{25} = 5; \ \sqrt{9} + \sqrt{16} = 3 + 4 = 7 \neq 5.$$

Practice 57

The first exercises for powers, logarithms, and root extraction should remain totally in the area of smaller numbers as has already become familiar during mental arithmetic practice. There are many varied oral and written exercises of the following kind that can be done: What is the exponent of 4 to get 64?

In written form this question would look like this: $4^? = 64$. Then, next to it, one should write the new form:? = $\log_4 64$ and the solution $\log_4 64 = 3$, since $4^3 = 64$.

In the same way, root extraction should be carefully and repeatedly practiced in both oral and written form.

Problems in which the *operations* must be found are very stimulating.

Example: Using different operations, connect the numbers 4, 5, and 625.

Solution: $5^4 = 625$, $\log_5 125 = 4$, $\sqrt[4]{625} = 5$.

1. Using as many different operations as possible, correlate the given numbers.

a)	*2, 3, 9*	*b)*	*8, 2, 64*	*c)*	*10, 3, 1000*
d)	*15, 10, 150*	*e)*	*120, 4, 30*	*f)*	*120, 4, 116*
g)	*2, 2, 4*	*h)*	*1, 1, 1*	*i)*	*1, 0, 0*
j)	*1, 1, 0*	*k)*	*0, 0, 0*	*l)*	*10, 10, 1*

Solutions:

a) $3^2 = 9$; $\sqrt[2]{9} = 3$; ${}_3log\,9 = 2$; b)

b) $8^2 = 64$; $\sqrt[2]{64} = 8$; ${}_8log\,64 = 2$;

c) c) $10^3 = 1000$; $\sqrt[3]{1000} = 10$; ${}_{10}log\,1000 = 3$;

d) $15 \cdot 10 = 150$; $150 : 10 = 15$; $150 : 15 = 10$;

e) $120 : 4 = 30$; $120 : 30 = 4$; $4 \cdot 30 = 120$;

f) $120 – 4 = 116$; $120 – 116 = 4$; $116 + 4 = 120$;

g) $2 + 2 = 4$; $2 \cdot 2 = 4$; $4 : 2 = 2$; $4 – 2 = 2$;

h) $1 \cdot 1 = 1$; $1 : 1 = 1$; $1^1 = 1$; ($\sqrt[1]{1} = 1$);

i) $1 \cdot 0 = 0$; $0 : 1 = 0$;

j) $1 – 1 = 0$; $1^0 = 1$;

k) $0 + 0 = 0$; $0 – 0 = 0$; $0 \cdot 0 = 0$;

l) $10 : 10 = 1$; $1 \cdot 10 = 10$; $10 : 10 = 1$.

The following group of problems should be done in a similar way.

2. Using one operation, correlate the given three numbers in such a way that the *third* number becomes the *result*.

a)	*81, 3→4*	*b)*	*25, 23→2*	*c)*	*25, 25→1*
d)	*125, 5→25*	*e)*	*125, 5→3*	*f)*	*125, 3→5*
g)	*125, 5→625*	*h)*	*125, 3→128*	*i)*	*64, 2→32*
j)	*64, 4→3*	*k)*	*64, 3→4*	*l)*	*64, 8→8*
m)	*8, 2→64*	*n)*	*64, 2→8*	*o)*	*64, 2→66*

Solutions:

a) ${}_3log\,81 = 4$; b) 25 – 23 = 2; c) 25 : 25 = 1; d) 125 : 5 = 25; e) ${}_5$log 125 = 3; f) $\sqrt[3]{125} = 5$; g) 125 · 5 = 625; h) 125 + 3 = 128; i) 64 : 2 = 32; j) ${}_4$log 64 = 3; k) $\sqrt[3]{64} = 4$; l) 64 : 8 = 8; m) 8^2 = 64; n) $\sqrt[2]{64} = 8$; o) 64 + 2 = 66.

I.5 More Correlations between Mathematical Operations

Just as we transitioned from addition into multiplication and further into powers, in the same way, we can use subtraction to lead into division and further into root extraction. If this is *gently* done, then it is more transparent and understandable for the students. It is essential that the correlation between mathematical operations is apparent and that root extraction is not limited to finding the square root, but rather is presented as a real mathematical operation using any root exponent. In my own 7th grade classes, I repeatedly went over all nine calculation types as they are presented in the volume *Der Anfangsunterricht in der Mathematik an Waldorfschulen.* In doing so, the students were exposed to logarithms at the same time as root extractions and powers. Even though these operations are not yet covered in any great detail, looking at logarithms at the same time as powers and roots is very relevant and does not cause any real difficulty if it is initially limited to whole-number relationships.

From Subtraction to Division

If we have two different (natural) numbers, we can subtract the smaller from the larger usually many times over, but at least once, until the remainder is either equal to zero or between zero and the smaller number. For example, think of a length that is 12 meters long. We can subtract a length of 5m two times from it. The remainder is 2m. If one can subtract a length multiple times without a remainder then we say that we have divided the length into equal pieces. For example, if we repeatedly subtract a 3m length from a 12m length, then after four subtractions we have no remainder. We say that we have divided a 12m length into 4 equal parts of 3m each.

A problem that asks to divide 18m into 3 equal parts means that we need to find the number that can be subtracted three times with no remainder. This problem can be written as follows:

$$3\begin{cases} 18m & - & = \\ & - & = \\ & - & = \ 0\,m \end{cases}$$

The original length of 18m is given, as well as the three-time subtraction and the result of 0m. What number can be subtracted so that the result really is 0?

Of course, we know that 6m is the correct answer. But if we calculated the problem using 5, for example, we would get:

$$18m - 5m = 13m$$
$$13m - 5m = 8m$$
$$8m - 5m = 3m > 0m$$

Since the result is *greater* than *0m*, the subtrahend (*5m*) is *too small*. If we choose a subtrahend that is too large, such as 7m, for instance, then the calculation leads into the negative:

$$18m - 7m = 11m$$
$$11m - 7m = 4m$$
$$4m - 7m = -3m < 0m$$

The end result is where we can tell if the subtrahend was too small or too large.

When the correct subtrahend is chosen (6m) we get:

$$18m - 6m = 12m$$
$$12m - 6m = 6m$$
$$6m - 6m = 0m$$

This calculation can also be written as follows, whereby we always have the beginning amount of 18m in view:

$$18m - 6m = 12m$$
$$18m - 6m - 6m = 6m$$
$$18m - 6m - 6m - 6m = 0m$$

This is another way of writing it:

$$18m - 1 \cdot 6m = 12m$$
$$18m - 2 \cdot 6m = 6m$$
$$18m - 3 \cdot 6m = 0m$$

The last equation shows that we have divided 18m into three equal parts of 6m each. In other words, we have divided 18 by 3.

$$\frac{18m}{3} = 6m$$

When dividing a number *a* into *n* equal parts, then one must find the number *b* that can be subtracted from *a* *n*-times with a remainder of 0. We can write this as follows, whereby it is understood that r_1, r_2, … is the remainder:

$$n \begin{cases} a - 1 \cdot b = r_1 \\ a - 2 \cdot b = r_2 \\ \ldots\ldots\ldots\ldots\ldots \\ a - n \cdot b = 0 \end{cases}$$

If a – n · b = 0, it means that:

$$\frac{a}{n} = b$$

Here, a series of exercises can be given in which the division is done as in the above examples; either in the first or second form.

Practice 58

1. Calculate by repeated subtraction:

a) 51 : 3; b) 76 : 4; c) 3,6 : 3; d) 4,2 : 7; e) 448 : 4; f) 14,4 : 3.

Solution:

1. a) 51 : 3 = 17, since 51 – 17 = 34
43 – 17 = 17
17 – 17 = 0
b) 76 : 4 = 19, since 76 – 19 = 57
57 – 19 = 38
38 – 19 = 19
19 – 19 = 0
c) 3,6 : 3 = 1,2, since 3,6 – 1,2 = 2,4
2,4 – 1,2 = 1,2
1,2 – 1,2 = 0
d) 4,2 : 7 = 0,6, since 4,2 – 0,6 = 3,6
3,6 – 0,6 = 3,0
3,0 – 0,6 = 2,4
2,4 – 0,6 = 1,8
1,8 – 0,6 = 1,2
1,2 – 0,6 = 0,6
0,6 – 0,6 = 0
e) 448 : 4 = 112, since 448 – 112 = 336
336 – 112 = 224
224 – 112 = 112
112 – 112 = 0
f) 14,4 : 3 = 4,8, since 14,4 – 4,8 = 9,6
9,6 – 4,8 = 4,8
4,8 – 4,8 = 6

Other written forms are possible, such as:

e) 3,6 – 1,2 = 2,4
3,6 – 2 · 1,2 = 1,2
3,6 – 3 · 1,2 = 0

Remarks:

Bringing division back through a process of repeated subtraction could obscure the autonomy of division. However, in exchange, a few simple exercises can keep one’s awareness sharp:

2. There is a given distance of several meters in length. Divide this distance with 6 (5, 4…) equal steps. Since one is only estimating the size of the steps when using small numbers, the distance must not be measured step for step, but rather the whole distance will be seen as an impressed structure – similar to the following experiment.

3. A relatively long rope is held by the teacher and one student or by two students. One of them begins to swing the rope so that knots begin to form at regular intervals in the rope. The entire length of the rope is thus divided; *a rhythm dividing*.

From Dividing to Root Extraction

In the introduction to powers we used the example of $3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$. From 81 how can we now get back to the root from which it was produced in four steps?[66] If a number is *multiplied* when raising to a given power, then on the way back it must be *divided*.

Finding the 4^{th} root of 81 means finding the number by which 81 is divided four times in order to get the result of 1.

The divisions that were carried out through multiple subtractions resulted in a zero at the end. But multiple divisions done to find the root must result in 1 at the end. The one is to multiplication and division what the zero is to addition and subtraction. They do not change the value of a number.

If one does the repeated divisions in the same order as the subtractions in the previous problem[67], one gets:

$$4 \left\{ \begin{array}{lcl} 81 & : & = \\ & : & = \\ & : & = \\ & : & = 1 \end{array} \right.$$

We know the *beginning* – the number 81 as radicand -, *the number of divisions* – the root exponent 4 -, and *the end result* of 1. What number is the divisor?

Let us first choose a number that is *too small*. The number 2, for example, would give us this with four divisions:[68]

$$\begin{array}{lccll} 81 & : & 2 & \approx\cdot & 40 \\ 40 & : & 2 & = & 20 \\ 20 & : & 2 & = & 10 \\ 10 & : & 2 & = & 5 > 1 \end{array}$$

Since after four divisions one number appeared that is *greater* than 1, the divisor, that is, the assumed root value, must be *too small*. If we check using the number 4 we get:

$$\begin{array}{rcccl} 81 & : & 4 & \approx & 20 \\ 20 & : & 4 & = & 5 \\ \frac{1}{36}\ 5 & : & 4 & = & 1{,}25 \\ 1{,}25 & : & 4 & \approx & 0{,}3 < 1 \end{array}$$

Since after four divisions the result is *less than* 1, the assumed root value must be *too large.* If we correctly choose the number 3 as the divisor we get:

$$\begin{array}{rcccr} 81 & : & 3 & = & 27 \\ 27 & : & 3 & = & 9 \\ 9 & : & 3 & = & 3 \\ 3 & : & 3 & = & 1 \end{array}$$

That is: $\sqrt[4]{81} = 3$

In this way a series of whole-number roots can be easily calculated. The following

[66] As has already been stated, one can also think of 81 as coming from *four* multiplications: $81 = 1 \cdot 3 \cdot 3 \cdot 3 \cdot 3$.

[67] It is important here to do parallel presentations on the blackboard of division through subtraction and root extraction through division. The students will understand the process because of the parallel presentation.

[68] We are only roughly calculating. The students should already be familiar with calculating with decimal fractions. Of course, one must keep in mind that approximate calculations are not always reliable, especially when miniscule differences play an important role.

exercises are presented as examples.

Practice 59

Calculate:

$$\sqrt[3]{8},\ \sqrt[3]{64},\ \sqrt[5]{32},\ \sqrt[3]{125},\ \sqrt[10]{1024},\ \sqrt[4]{625},\ \sqrt[2]{64},\ \sqrt[3]{64},\ \sqrt[6]{64},\ \sqrt[2]{49},\ \sqrt[3]{343}$$

Solutions: 2; 4; 2; 5; 2; 5; 8; 4; 2; 7; 7.

In the next step root extraction can be brought back to division:

If the square root of the number *a* is to be found, then the number *b* must be found that in two steps of division of *a* results in 1. In the following, q_1 and q_2 designate the successive quotients that are the result of the divisions.

$$2\ \begin{cases} a : b = q_1 \\ q_1 : b = 1 \end{cases}$$

On the last line we see that in this case it must be that $q_1 = b$. We could also write:

$$2\ \begin{cases} a : b = b \\ b : b = 1 \end{cases} \quad \text{or} \quad 2\ \begin{cases} a : b = a : b \\ (a : b) : b = a : b^2 = 1 \end{cases}$$

In the last form it is apparent that $a = b^2$. Therefore, b is $\sqrt[2]{a}$.

Important tip: Since the square root appears especially often in mathematics, it has been agreed that in this case the root exponent must not be written. If one finds that a root is given without a root exponent it means that it is a square root: $\sqrt{a} = \sqrt[2]{a}$. One must pay careful attention when using calculators.

If the cubic root of *a* is to be found, then we must find a *b* that results in the number 1 after three steps of divisions:

$$3\ \begin{cases} a : b = q_1 \\ q_1 : b = q_2 \\ q_2 : b = 1 \end{cases}$$

This time it must be that $q_2 = b$. We could also write:

$$3\ \begin{cases} a : b = q_1 \\ q_1 : b = b \\ b : b = 1 \end{cases}$$

In the same way as the square root, we could also write:

$$3\ \begin{cases} a : b = a : b \\ (a : b) : b = a : b^2 \\ (a : b^2) : b = a : b^3 = 1 \end{cases}$$

From this we read: $b = \sqrt[3]{a}$

If we are to find the *nth* root of *a,* then we must find a *b* that in *n* divisions of *a* results in the number 1:

$$n\ \begin{cases} a : b = q_1 \\ q_1 : b = q_2 \\ \quad \ldots\ldots\ldots\ldots \\ b : b = 1 \end{cases}$$

With that we have led the process of root extraction back to division.

When we learn to use indices as in the chapter on recursive arithmetic, we can also write it like this:

$$n\begin{cases} a : b = q_1 \\ q_1 : b = q_2 \\ \dots\dots\dots\dots \\ q_{n-2} : b = q_{n-1} \\ q_{n-1} : b = 1 \end{cases}$$

In order to have this: $q_{n-1} : b = 1$, one must, again, have this: $q_{n-1} = b$

Just as with the square and cubic root, one can also write it like this:

$$n\begin{cases} a : b = a : b \\ (a : b) : b = a : b^2 \\ (a : b^2) : b = a : b^3 \\ \dots\dots\dots\dots \\ (a : b^{n-2}) : b = a : b^{n-1} \\ (a : b^{n-1}) : b = a : b^n = 1 \end{cases}$$

From this follows: $b = \sqrt[n]{a}$

Bringing root extraction back to division can also be helpful to us in cases where we cannot exactly determine the root, but wish to estimate it.

For example, one is trying to find the cubic root of 100, we do the calculation as follows:

$$3\begin{cases} 100 : \quad = \\ \quad : \quad = \\ \quad : \quad = 1 \end{cases}$$

If we choose the number 4 as the assumed root value, we get this:

$$3\begin{cases} 100 : 4 = 25 \\ 25 : 4 = 6{,}25 \\ 6{,}25 : 4 > 1 \end{cases}$$

The number 4 is *too small*. If we choose the number 5, we get:

$$3\begin{cases} 100 : 5 = 20 \\ 20 : 5 = 4 \\ 4 : 5 < 1 \end{cases}$$

The number 5 is too large. From these two calculations we learn something: When writing in decimal form the root must be written with a 4 in front of the decimal point.

If we now check the decimal fraction values, we can always determine if they are too large or too small. In our case, 4.6 results in:

$$\begin{aligned} 100 : 4{,}6 &\approx 21{,}7 \\ 21{,}7 : 4{,}6 &\approx 4{,}7 \\ 4{,}7 : 4{,}6 &> 1 \end{aligned}$$

4.6 is too small. When we check 4.7 we get:

$$\begin{aligned} 100 : 4{,}7 &\approx 21{,}3 \\ 21{,}3 : 4{,}7 &\approx 4{,}5 \\ 4{,}5 : 4{,}7 &< 1 \end{aligned}$$

4.7 is *too large*. Coming closer to the root value requires that a 6 come after the decimal point:

$$\sqrt[3]{100} = 4.6$$

In the next step one finds:

$$100 : 4{,}64 \approx 21{,}55$$
$$21{,}55 : 4{,}64 \approx 4{,}645$$
$$4{,}645 : 4{,}64 > 1$$

4.64 is *too small*, so one must look at more numbers after the decimal point to find the correct number. If we check 4.65 we get:

$$100 : 4{,}65 \approx 21{,}505$$
$$21{,}505 : 4{,}65 \approx 4{,}625$$
$$4{,}625 : 4{,}65 < 1$$

4.65 is too large. The next number to be chosen is 4:

$$^3\sqrt{100} = 4.64$$

If we check this answer by calculating the exponent we get:

$$4.64^3 = 99.897344$$

In the third power there remains a deficit of about 1/10th.

In principle, we could continue dividing in this way to get closer and closer to the root value. It is also possible, with the help of a simple calculator that has no root function, to determine any root exactly up to a few numbers after the decimal point.

In general, can we ever exactly determine any root?

Now, the question arises about *irrational numbers*. One can explain to the students most root values can *never* be completely and exactly represented by decimal fractions or normal fractions. Perhaps the easiest example of this is $^2\sqrt{2}$. The number two must be divided two times by what number *d* in order to get the number 1?

$$2 : d = d$$

$$d : d = 1$$

The students can try for a long time. In the end, one can give them the answer:

$$2\sqrt{2} = 1.414213562$$

If one multiplies this approximate value with itself, one still does not really get the number 2 as the result. Is it even possible to multiply a number with a finite number of decimal places with itself and get a whole number as a result? After some thought, this will appear to the students to be very unlikely.

The strict proof that $^2\sqrt{2}$ cannot be precisely solved through a decimal fraction with a finite number of decimal places or through a normal fraction, is too challenging for most students. This is brought up in a later class. For now, it can only be reported upon. Amounts that are represented by $^2\sqrt{2}$, for example, but do not result in a whole number or a fraction, are called *irrational*. The Greeks were deeply shaken by the discovery of the irrationals. The following has been handed down to us from Euclid: "There is a story that comes from the followers of Pythagoras which says that the first one to divulge the theory (of irrationals) in public was a victim of a shipwreck, and perhaps they wanted to point out with this story that everything irrational in the universe should remain as something "unspeakable and formless", and that when someone and their soul meets such a form of life, and makes it accessible and public, he is sucked into the ocean of becoming, and, from then on, will never experience the lapping of quiet currents."[69]

[69] Quoted from Louis Locher-Ernst, *Raum und Gegenraum*, Dornach ^{2}1970, Page 212. A worthwhile look at the irrationality of $\sqrt{2}$ is found in *Vom Denken in Begriffen. Mathematick als Experiment des reinen Denkens* by

Practice 60

Between what successive whole numbers are the given root values?

1. $\sqrt{10}$; $\sqrt{20}$; $\sqrt{30}$; $\sqrt{50}$; $\sqrt{70}$; $\sqrt{80}$; $\sqrt{90}$; $\sqrt{200}$; $\sqrt{500}$; $\sqrt{1000}$; $\sqrt{120{,}83}$; $\sqrt{930{,}2}$; $\sqrt{5{,}678}$.
2. $\sqrt[3]{10}$; $\sqrt[3]{20}$; $\sqrt[3]{30}$; $\sqrt[3]{90}$; $\sqrt[3]{700}$; $\sqrt[3]{10000}$.
3. $\sqrt[4]{10}$; $\sqrt[4]{20}$; $\sqrt[4]{30}$; $\sqrt[4]{100}$; $\sqrt[4]{300}$;$\sqrt[4]{2000}$.

Solutions:

1. $3 < \sqrt{10} < 4$; $4 < \sqrt{20} < 5$; $5 < \sqrt{30} < 6$; $7 < \sqrt{50} < 8$; $8 < \sqrt{70} < 9$;
$8 < \sqrt{80} < 9$;$9 < \sqrt{90} < 10$; $14 < \sqrt{200} < 15$; $22 < \sqrt{500} < 23$;
$31 < \sqrt{1000} < 32$; $10 < \sqrt{120{,}83} < 11$; $30 < \sqrt{930{,}2} < 31$; $2 < \sqrt{5{,}678} < 3$.
2. $2 < \sqrt[3]{10} < 3$; $2 < \sqrt[3]{20} < 3$; $3 < \sqrt[3]{30} < 4$; $4 < \sqrt[3]{90} < 5$; $8 < \sqrt[3]{700} < 9$; $\sqrt[3]{10000} = 100$.
3. $1 < \sqrt[4]{10} < 2$; $2 < \sqrt[4]{20} < 3$; $3 < \sqrt[4]{100} < 4$; $4 < \sqrt[4]{300} < 5$; $6 < \sqrt[4]{2000} < 7$.

II Calculating a Square Root

I.6 (Should be II.6 etc.) Preliminary Remarks

In the following we will look at a long- known arithmetic method for determining a square root. Being able to understand and use this method – one speaks of an algorithm – above all, has intrinsic educational value. First, one looks at the interplay of the different operations, then an expedient way of estimating is applied, and finally, algorithms are an essential part of computer programs. The objection that such methods are superfluous since the invention of computers because they can compute every root, not just square roots, with enough accuracy, is correct in terms of practical, professional use. To understand how such algorithms work is, in general, not really explained later. In any case, most students will enjoy learning how they work and how they are used.

Rudolf Steiner asked a mathematics teacher at the opening of a 9th grade class on September 22, 1920, how he had gone about teaching powers and root extraction in the previous 8th grade; especially how he had taught squaring and cubing of certain numbers, as well as finding the roots, including cubic roots; and he said: “With these things it does not matter so much that one does them in the same way as they will be used later, but rather that certain forms of thinking are practiced. The thinking forms that one uses with cubing, squaring, and root extraction, this singularity, that is, in a certain way, abstracted from the concreteness of numbers, and then newly grouping the numbers in other ways, leads so deeply into the structure of numbers, and is so formative for thinking, that one has to do it.”[70]

Those who really delve into the individual steps of the method will understand Rudolf Steiner’s viewpoint.

However, I am not of the opinion that everything presented here must be taught. Every teacher will best be able to judge for themselves where the limits will be placed. It appears to me that the first levels of root algorithms can be directly taught in the classroom. However, an important question is just how far one can go in explaining this arithmetic method to the students. The formative aspect of which Rudolf Steiner was speaking does not lie in mastering a skill, but rather in the special application of the binomial formula.

Alexander Israel Wittenberg, Basel and Stuttgart, 1957.

[70] Rudolf Steiner, *Konferenzen mit den Lehrern der Freien Waldorfschulen 1919 to 1924,* volume I, Complete Works 300a, Page 221. See also: Detlev Hardorp, *Erziehung zur Knechtschaft – oder zur Freiheit. Zur menschenbildenden Wirkung des Mathematikunterrichts am Beispiel des Wurzelziehens.* In: *Erziehungskunst, Oct. 1996.*

My question is this: To what extent can the inner composition of root algorithms be made so understandable for the most capable students that the entire class becomes familiar with them?

It is an illusion to think that every student will be able to follow without any problem. But, in every class, the teaching should rightly occur with different levels of ability in mind. Just one well-presented and developed thought from the group of most able students is significant for the whole class and creates trust in the use of algorithms. Experiencing such discussions has great significance for later intellectual development of those students who are weaker in mathematics. They have a stimulating effect on one's own development. I often experienced this while working with colleagues who had had a hard time with mathematics as children. However, later, because they experienced it with their students, understanding was easily gained.

Preparations

With the previous calculations of an exact or approximate root value, it remained unsatisfying that the search was so unsystematic. For this reason, mathematicians have thought long and hard about the most effective way of finding a root value. There have been a series of various methods discovered, a few of which will be discussed in the following pages. It is important to have knowledge of basic forms of arithmetic, the first square numbers, and the first binomial formula that we have already gone over. It has to do with developing a method that will allow the calculation of *every* square root to the desired accuracy for practical use, and not just the easy cases.

II Necessary Tools

1. The essential basis for the method we want to develop is the first binomial formula

$$(a + b)^2 = (a + b) \cdot (a + b) = a^2 + 2ab + b^2$$

In this way it is possible for products of equal sums on the left side $(a + b) \cdot (a + b)$ to be converted into a sum of products $a^2 + 2ab + b^2$.

2. If we compare the series of natural numbers with their squares, we find that with increasing numbers, the squares stand in increasing ratios to the base numbers. 1 and 1^2 are still the same, but 2 is only half of 2^2, and 100 is only 1% of its square. n^2 is n-times greater than n.

From this we come to an interesting conclusion that will be useful to us. We ask: How is the square of a number *a* changed if we increase it a little (by the smaller number *b*) and go from a^2 to $a + b$ and

$$(a + b)^2?$$

Example:

First, let us start with: a = b = 10

$$(10 + 10)^2 = 10^2 + 2 \cdot 10 \cdot 10 + 10^2 = 100 + 200 + 100 = 400$$

Of course, *a* and *b* contribute equally to the result. The mixed term 2*ab* is just as large as both squares together.

Now, we increase *a* and correspondingly decrease *b* so that always $a + b = 20$. We will get this series:

$$(10 + 10)^2 = 100 + 200 + 100 = 400$$

$$(11 + 9)^2 = 121 + 198 + \ \ 81 = 400.$$

$(12 + 8)^2 = 144 + 192 + 64 = 400.$

$(13 + 7)^2 = 169 + 182 + 49 = 400.$

$(14 + 6)^2 = 196 + 168 + 36 = 400.$

$(15 + 5)^2 = 225 + 150 + 25 = 400.$

$(16 + 4)^2 = 256 + 128 + 16 = 400.$

$(17 + 3)^2 = 289 + 102 + 9 = 400.$

$(18 + 2)^2 = 324 + 72 + 4 = 400.$

$(19 + 1)^2 = 361 + 38 + 1 = 400.$

$(20 + 0)^2 = 400 + 0 + 0 = 400.$

One sees: The three terms a^2, 2ab, and b^2 contribute very differently to the result depending upon their size. If b is much less than a, one writes $b \ll a$. In this case the change in a^2 is mainly determined by the term $2ab$. b^2 plays a subordinate role. If one wishes to only estimate the increase in a square when there is a slight increase in the base number, then it is enough to take the binomial formula into consideration with a^2 and the middle term $2ab$:

If b is much less than a ($b \ll a$), then

$$(a + b)^2 = a^2 + 2ab.$$

Admittedly: The less the difference is between a and b, the larger will b^2 be in proportion to a^2, and must therefore be all the more quickly taken into consideration. This ability to estimate will be of good service to us.

III Calculating some Roots

Example 1:

In order to understand the following method of calculating a square root we will choose a problem that is easy to solve. Let us assume we want to calculate √676, but we do not know the result immediately. Let us forget for a moment that we know from the series of square numbers that:

676 = 262, that is, √676 = 26.

First, we will decide how many digits in the result go before the decimal point. To do this, we turn everything around: If there is one digit in front of the decimal point, then the square is one or two digits. If there is a two-digit number, then the square has three or four digits, and so on. Therefore, the root of a one or two digit number must have one digit, of a three or four digit number two digits, and so forth. A table can be of help:

x	$\mathbf{x^2}$
1 .... 9,...	1 .... 99,...
10 .... 99,...	100 .... 9999,...
100 .... 999,...	10.000 .... 999999,...

Since 676 is three digits, √676 must have two digits in front of the decimal. First, we mark the number of digits in front of the decimal with points:

√676 = . .

(That the points are actually not needed later is something we choose to forget for right now.)

Now we will consider what the first digit (in the tens column) of the result must be. We know that $20^2 = 400$ and $30^2 = 900$. Because the number 676 lies between these numbers, the root will also be between 20 and 30. In any case, the first digit must be 2. 20 is a first approximation of the root value we are trying to find, whereby we are certain that the 2 in the tens column is correct. The number 20 is considered the *a* in the above binomial formula for estimations. We now want to find a number *b* (which must be smaller than 10 since the first digit may not be changed), which will adjust *a*, since a^2 still does not give the desired number 676.

What amount should be chosen for *b* so that:

$676 = (20 + b)^2$?

We square it out and get the following:

$676 = 400 + 40b + b^2$

By subtracting 400 from both sides we get:

$276 = 40b + b^2 = (40 + b) \cdot b$

With this it is no easier to calculate *b* than it was before to calculate the root directly from 676. But remember that b is a little smaller than a and therefore, because of the said b^2 in proportion to the middle digit 2ab = 40b, cannot play a very big role. So, we take b^2 out of consideration for a moment and estimate the amount of *b*:

$276 = 40b$

Since we are, for the moment, only interested in the digit in the units' column, we will estimate in whole numbers:

$B = 276 : 40 = 27 : 4 = 6$

Does this fulfill our condition?

$$276 = 40b + b^2 = (40 + b) \cdot b \qquad (*)$$

If we insert b = 6 we get:

$(40 + b) \cdot b = (40 + 6) \cdot 6 = 240 + 36 = 276$

The condition (*) is fulfilled, and we have gained the correct result at the same time: The necessary adjustment of a = 20 is b = 6, which is:

$\sqrt{676} = a + b = 26$

We will calculate and check some further examples using the same method:

Example 2:

Find $\sqrt{961}$

Again, the result will be a two digit number:

$\sqrt{961} = ..$

The first digit of the result must be 3 since 961 lies between $900 = 30^2$ and $1600 = 40^2$. We choose

a = 30 as the first approximation of the root value.

How do we get the correction b so that a + b is two correct digits?

It should be:

$(a + b)^2 = (30 + b)^2 = 961 \quad or\ 900 + 60b + b^2 = 961$

Or rather

$60\ b + b^2 = (60 + b) \cdot b = 61$

Again, if we ignore b^2 for a moment, we get the approximate correlation:

$60b = 61$

The possibilities for b are the numbers 0 or 1 since, considering the b^2, we must always make it somewhat smaller. If we put in b = 1, we get:

$(60 + b) \cdot = (60 + 1) \cdot 1 = 61$

b = 1 fulfills the required condition exactly and it is:

$\sqrt{961} = a + b = 31$

Example 3:

Find √3969:

The result must have two digits in front of the decimal point. The last two digits, 69, have no significance for the first digit of the root value. Instead of asking between what squares of a full tens column digit does 3969 lie, it is enough to ask: Between what square numbers does 39 lie? What matters is the largest whole number whose square is either less than or equal to 39. This is the number 6. The first digit is 6, and our first approximation is a = 60. How large is the necessary adjustment b?

The condition for b is:

$(a + b)^2 = (60 + b)^2 = 3969$

Through squaring and simplifying we get:

$120b + b^2 = (120 + b) \cdot b = 369$

This approximation serves to estimate b:

$120b = 369$

From this we see that cannot be greater than 3. b = 3 leads to:

$(120 + 3) \cdot 3 = 123 \cdot 3 = 369$

So that the condition is fulfilled exactly:

$\sqrt{3969} = a + b = 63$

Example 4:

Find √729

The root must have two digits in front of the decimal point. The tens column digit must be 2 because:

$20^2 < 729 < 30^2 \ \ or\ rather\ \ 2^2 < 7 < 3^2$

Our first approximation is a = 20. How large is the necessary adjustment b?

The condition for b is:

$(a + b)^2 = (20 + b)^2 = 729$

By squaring and converting we get:

$(40 + b) \cdot b = 329$

To estimate b we ignore the additive b in parentheses for the moment and we get:

$40b = 329$

From this we get:

$b = 8$

Here, we must be careful because, in proportion to 20, 8 is not so small that it can easily remain unconsidered. Let us check what value the expression $(40 + b) \cdot b$ has when $b = 8$. We then get:

$(40 + 8) \cdot 8 = 384 > 329$

8 is therefore too large. So, we now check the next smaller number, 7. With 7 we get:

$(40 + 7) \cdot 7 = 329$

Now the condition has been fulfilled exactly. The result is:

$\sqrt{729} = 27$

This example is intended to show that the estimation through which the b^2, or the additive *b* in parentheses, is left out, can lead to a *b* that is too large. The main thing is to be careful when a and b are not very different.

Up to now the examples have been chosen so that *b* exactly fulfills the conditions. In order to expand our experience, we will look again at a "well-behaved" example which requires the calculation of more than two digits:

Example 5:

Find the root value for $\sqrt{18671041}$.

The number of digits in front of the decimal point in the result can be determined by starting with a units column digit and forming groups of two that are separated by '. That is, write √18'67'10'41 and count the number of groups. The first *number group* on the left can also consist of only 1 digit. The number of groups equals the number of digits in front of the decimal point in the result. In our example the result has four digits in front of the decimal point.

In this example we will consider principles as well as a rational organization of the calculations.

We prepare to do the actual calculation of √18'67'10'41 by marking the four digits in front of the decimal point like this:

√18'67'10'41 = ...

Just as in the previous example, we find the digit in the first position by finding the largest number whose square is less than or equal to the first "number group" (18). In our example it is the number 4, since $4^2 < 18 < 5^2$.

We now have:

√18'67'10'41 = 4 . . .

The first approximation of the root value is a = 4000. Now, as before, we look for an adjustment, b, that adequately fulfills the following condition:

$(a + b)^2 = (4000 + b)^2 = 18\ 67\ 10\ 41$

Or, squared and simplified:

$8000\ b + b^2 = (8000 + b) \cdot b = 2\ 67\ 10\ 41$

With the last conversion a2 = 16000000 was already removed from both sides. We note that until now only the first number group on the left is actually used in the calculation.

Shortened, we do the subtraction in the following way and ignore the other number groups for now:

$$\begin{array}{l} \sqrt{18'67'10'41} = 4\ .\ .\ .\ , \\ \underline{16} \\ \ \ 2 \end{array}$$

From the adjustment term b, we first only require that we gain another digit for the root value. In a certain way, limiting our requirement justifies the rough method of estimation we are using. For the time being, we ignore the b^2 in $800b + b^2$, or rather, the additive b in its converted form of $(8000 + b) \cdot b$. We get:

$8000\,b = 2671041$ *or* $b = 2671041 : 8000$

Since we are only looking to find the digit in the first number column with the quotient, the last digits of the calculated difference of 2’67’10’41 do not play a role. It is:

$b = 2671041 : 8000 = 2671 : 8 = (26 : 8) \cdot 100 = 3 \cdot 100$

The last conversion is done because we are calculating only a single-digit number that we automatically write in the hundreds column (second from the left) thereby giving it 100 times the value.

In the short form of algorithms, the calculation has the following form:

After the subtraction, the next “number group” (67) is “brought down” and written on the right next to the difference 2. The last digit (7) remains unconsidered for now. One writes 26’7.

Now, 26 will be divided by 8, as has been done above in the parentheses. 8 is the doubled 2a of the first approximation, a, without considering the place value. (More precisely: 2a is 8000, but we do not want to deal with the zeros.) What we get is:

$$\begin{array}{lll} \sqrt{18'67'10'41} = 4\ .\ .\ .\ , & & \\ \underline{16} & & \\ 26'7 & :8 & (\approx 3.\ \textit{This is not yet written down}!) \end{array}$$

Now, we will go a step further: After the estimation of

$b \approx 2671041 : 8000 \approx (26 : 8) \cdot 100 \approx 3 \cdot 100,$

we must first make sure that b is not too large, and what difference remains between the radicand and

$(a + b)^2 = (4000 + 300)^2 = 4300^2.$

With b inserted into

$(8000 + b) \cdot b$ *we get* $(8000 + 300) \cdot 300 = 8300 \cdot 300 = 2490000$

And, from the previous remainder (2671041) we get the difference

$$\begin{array}{r} 2671041 \\ -\ \underline{2490000} \\ 181041 \end{array}$$

In the short form this calculation becomes the following: The result of the division (3) is written three times; to the right of the result beside the already calculated digit 4, below right next to the divisor (8) (The subscript positioning is done so that one can recognize that it has been added; this important later when checking the answer, but what is meant if the formation of the number 83; that is, 8_3= 83, and once again to the right next to the multiplier:

$$\sqrt{18'67'10'41} = 43\ .\ .\ .\ ,$$
$$\begin{array}{l} \underline{16} \\ \ 26'7 \qquad : 8_3 \cdot 3 \end{array}$$

While forming 8_3 we add

$a + b = 8000 + 300 = 8300$

without carrying bothersome zeros.

If we do the multiplication and write the result under 267, then we can again quickly calculate the difference – as long as it is still of interest at this stage: Instead of 181041, we are happy with the first two digits of 18 that are then expanded to 1810 through the next number group (10).

$$\sqrt{18'67'10'41} = 43\ .\ .\ .\ ,$$
$$\begin{array}{l} \underline{16} \\ \ 26'7 : 8_3 \cdot 3 \\ \underline{24\ 9} \\ \ \ 1\ 810 \end{array}$$

Now, in principle, the calculation can proceed in the same way. However, before we do that, we want to look back once again and clarify what has been found: In the first step, from our knowledge of square numbers, we calculated the first digit, 4, as the whole number root from the left-side number group, 18. This gave us the first *a*. Then we approximated an adjustment term *b*, that improved the *a* in the second position on the left. But since this adjustment did not yet lead to the exact value, we began the operations from the front, except that we started with the improved value of a_1 = 4300 instead of a = 4000. We are trying to find the adjustment term b_1 so that $(a_1 + b_1)$ comes close enough to the radicand.

Again, b_1 should provide only one further number column; the third from the left. Just as *b* in the middle is less than *a* by a factor of 10, so is b_1 less than *b*. If we compare (a + b) and $(a_1 + b_1)$, we see that the left summand (a) has increased while the adjusting term (b) has decreased. In this way we continually increase the exactness. We should determine b_1 so that

$(a_1 + b_1)^2 = (4300 + b_1)^2$

comes close enough to 18’67’10’41 but is not greater than this number. According to the binomial formula and our experience up to now, this leads to:

$2a_1b_1 + b_1^2 = (2 \cdot 4300 + b_1) \cdot b_1 = (8600 + b_1) \cdot b_1 = 181041$

But instead of calculating a_1^2 anew and subtracting it from 18’67’10’41, we can go back to the previous calculations because the difference is there already.

We get the short form of the estimation of b_1 in the same way as before: For right now, we will ignore the last digit (0) in 1810, which we will signify with a ‘, and divide by double a_1 (without considering the zeros to the right of the already calculated number columns). Using the same notation as above, we get:

$$\sqrt{18'67'10'41} = 432\ .\ .\ ,$$
$$\begin{array}{lll} \underline{16} & & \\ 26'7 & : & 8_3 \cdot 3 \\ \underline{24\ 9} & & \\ \ \ 1\ 81'0 & : & 86_2 \cdot 2 \\ \ \ \underline{1\ 72\ 4} & & \\ \qquad 8\ 6 & & \end{array}$$

From this we have now found the number 2 in the third column. 20 is $b_{1.}$ If we bat this over to $a_{1,}$ then we again get the increased value $a_2 = a_1 + b_1 = 4320$.

But since there is a remainder we must try and find a further adjustment b_2 from a_2 that will determine the fourth column from the left. Again, the condition is: $(a_2 + b_2)^2 = (4320 + b_2)^2$ and should come as close as possible to the radicand without going over it.

If $(a_2 + b_2)^2 = (4320 + b_2)^2$ is squared and the condition given that this expression should come as close as possible to 18’67’10’41, then during simplification all the subtractions already done have accrued and it remains:

$$2a_2b_2 + b_2^2 = b_2 \cdot (2a_2 + b_2) = 8641$$

For the estimation we again use:

$$b_2 \cdot 2a_2 \approx 8641$$

$$b_2 \approx 8641 : 2a_2$$

$2a_2$ is double the previous results. If we again ignore the zeros in the columns that have not been calculated yet, then, for the moment, we must also leave the last digit (1) of the remainder 8641 out of consideration. The estimation occurs out of:

$$b_2 = 864'1 : 864 = 1$$

The improved value is:

$$a_3 = a_2 + b_2 = 4320 + 1 = 4321$$

In the algorithm short form this calculation is done as before: The last number group (41) is brought down, written on the right next to the remainder (86), and the last digit (1) is separated with a ‘. Then it is divided by double the existing results 2 · 432 = 864. The result is noted on three number columns and the difference is calculated:

$$
\begin{array}{lllll}
\sqrt{18'67'10'41} = 4321 \\
\underline{16} \\
\ \ 26'7 & : & 8_3 & \cdot & 3 \\
\ \ \underline{24\ 9} \\
\ \ \ 1\ 81'0 & : & 86_2 & \cdot & 2 \\
\ \ \ \underline{1\ 72\ 4} \\
\ \ \ \ \ \ 864'1 & : & 864_1 & \cdot & 1 \\
\ \ \ \ \ \ \underline{864\ 1} \\
\ \ \ \ \ \ \ \ \ \ 0
\end{array}
$$

The last difference is zero. With that, the root value was exactly calculated:

$$\sqrt{18671041} = 4321$$

To check the answer we square it and get:

$$4321^2 = 18671041$$

If we look at the entire process of the calculation, then we see that the main point was to go as far as possible with the root extraction using mathematical operations that were as simple as possible. This was successful with the exception of the first step.

The basis for algorithms forms the binomial formula:

$$(a + b)^2 = a^2 + 2ab + b^2$$

It does this by converting the square of a sum into a sum of squares, or rather, products. The conversion of the sequence of operations makes it possible. The numbers are configured differently in relation to the operations.

In (a + b), *a* is always the derived estimation, and *b* is the adjustment for the next, improved approximate value. There is an increase in *a*, while *b* always decreases in relation to *a*, so that leaving *b2* unconsidered in the estimation is less and less "dangerous".

On the whole, finding the numbers using a **denominational number system** plays an important role, since we calculate to find only one further digit in every "round of calculations". In principle, the method is not limited to the decimal system, but is relatively easy to carry over to other systems. For practice, and for getting deeper into the subject, this can happen in the 9th school year when the **denominational number system** is studied further. **(Trans. Note: the dictionary gave the phrase denominational number system for the word Stellenwertsystem. I'm not convinced that's the right usage but I can't think of anything else.)**

Example 6:

In order to really impress this method on our memory we will use it again in another example; this time without explanatory remarks. In the process, at the same time, we will add the decimal places in the radicand since the method can be easily expanded to include that.

This is to be calculated: $\sqrt{147166156{,}064}$

The successive steps for finding the solution are:

1. Preparation

Starting at the decimal point, form two-number groups to the left and right and separate them with '. The first group to the left may consist of one or two digits. On the right, a zero, of necessity, will be supplemented so that each group there contains two digits. We prepare for the result by putting the same number of points in front of the decimal point as there are radicand number groups to the left of the decimal point.

$$\sqrt{1'47'16'61'56{,}06'40} = \;.\;.\;.\;.\;.\;.\;,$$

2. First approximation

We are looking for the largest whole number whose square is less than or equal to the first number group (1). This is the number 1. It is written in the first number column of the result and its square

($1^2 = 1$) is subtracted from the first number group:

$$\begin{array}{l}\sqrt{1'47'16'61'56{,}06'40} = 1\;.\;.\;.\;.\;.\;,\\ \underline{1}\\ 0\end{array}$$

3. Begin the repetition of calculations

Now, the next number group is brought down (47) and written next to the remainder (0). The last digit (7) is ignored for the time being and is separated by a '. The number in front of it (4) will be divided by double the result that is already there ($2 \cdot 1 = 2$). The result (2) is noted in three places: on the right next to the existing result, as subscript to the right of the divisor, and on the right of the divisor as the multiplier. The multiplication is done and the product on the left is subtracted:

$$\begin{array}{ll}\sqrt{1'47'16'61'56{,}06'40} = 12\;.\;.\;.\;.\;, & \\ \underline{1} & \\ 0\;4'7 & :\; 2_2 \cdot 2\\ \;\;\underline{4\;4} & \\ \;\;\;\;\;3 & \end{array}$$

4. Repetition of the calculation

The method is continued until an adequately exact root value is achieved. If there are no more number groups to the right of the decimal point then any amount of pairs of zeros should be used to supplement. This is the calculation:

$$
\begin{array}{ll}
\sqrt{1'47'16'61'56,06'40} = 12\,131,205 & \\
1 & \\
0\;4'7 & :\ 2_{2}\cdot 2 \\
\;\;4\;4 & \\
\;\;\;\;31'6 & :\ 24_{1}\cdot 1 \\
\;\;\;\;24\;1 & \\
\;\;\;\;\;\;7\;56'1 & :\ 242_{3}\cdot 3 \\
\;\;\;\;\;\;7\;26\;9 & \\
\;\;\;\;\;\;\;\;29\;25'6 & :\ 2426_{1}\cdot 1 \\
\;\;\;\;\;\;\;\;24\;26\;1 & \\
\;\;\;\;\;\;\;\;\;\;4\;99\;50'6 & :\ 24262_{2}\cdot 2 \\
\;\;\;\;\;\;\;\;\;\;4\;85\;24\;4 & \\
\;\;\;\;\;\;\;\;\;\;\;\;14\;26\;24'000 & :\ 242624_{05}\cdot 5 \\
\;\;\;\;\;\;\;\;\;\;\;\;12\;13\;12\;025 & \\
\;\;\;\;\;\;\;\;\;\;\;\;\;\;2\;13\;11\;975 &
\end{array}
$$

5. Ending the calculations

In most cases the root can not be completely determined as a decimal fraction. That means that the conversion of the defined number $z = \sqrt{R}$ into a sum, with the help of the root operation, can only give an approximate value. At lower levels this is already apparent because even 1/3 can not be converted into a (finite) decimal fraction:

1/3 = 0.333…

While here the reason for this is the relationship of the denominator 3 to the base number 10, actually, the inability to convert the roots has deeper causes; namely how the *operations* correlate with one another.[71]

However, in principle, the developed method allows for any *approximation* of the root value. With each calculated number column, the accuracy is raised by a power of 10. So, if we have the approximate value of:

$$\sqrt{147166156,064} \approx 12131,20,$$

then we know with certainty that the root allows for the following limit:

$$12131,20 \leq \sqrt{147166156,064} \leq 12131,21.$$

The uncertainty amounts to $\frac{1}{100}$.

Calculating the next number column:

$$\sqrt{147166156,064} \approx 12131,205$$

brings the uncertainty to $\frac{1}{1000}$ because the root value now allows the limit:

[71] Those who are interested in finding roots in ways other than decimal fractions are directed to the chain fractions. Compare, for example, Louis Locher-Ernst, *Arithmetik und Algebra,* Page 294.

$$12131{,}205 < \sqrt{147166156{,}064} < 12131{,}206$$

and so on.

When extracting roots from empirical amounts, as with all other calculation methods, one must decide how far to calculate the results and still consider it meaningful. If one were to determine something like the diagonal length *d* of a rectangular board with the side lengths of a = 100cm and b = 40cm using the Pythagorean Theorem as:

$$d = \sqrt{100^2 + 40^2}\ cm = \sqrt{11600}\ cm$$

Then, if we calculate the root up to 8 decimal places, the result is meaningless for all practical purposes because, among other things, the board's fluctuation in length caused by humidity and temperature changes is much greater than the calculated "exactness".[72]

One has to differentiate between the relationship of mathematical concepts (for which no imprecise definitions may exist) and the mathematical description of the outer world of experience. As opposed to mathematics, this experiential world, when it has to do with measured amounts, basically, must always develop a description that is limited.

Unless it comes out even, the method of calculating the root must be discontinued after a specified number of steps. In our example we calculated up to three decimal places. With the calculation of the fourth decimal place it can be decided if the third decimal place must be rounded up. We get:

$$\sqrt{147166156{,}064} = 12131{,}2058$$

The best approximation with three decimal places is:

$$\sqrt{147166156{,}064} \approx 12131{,}206$$

If the approximate number is squared, we get:

$$12131{,}206^2 \approx 147166159{,}0.$$

This means that when the 8 given places of the root are squared there is a deviation at the ninth place of the radicand.

IV More Examples, Special Cases

Through the use of a few more examples, we will solidify our expertise with the developed method, and, also, make a few remarks about questions that could possibly arise.

Example 7:

Calculate $\sqrt{5}$ up to the fifth decimal place.

There is a *single*-digit *number group* in front of the decimal point (5). The result also has *one* number place in front of the decimal. Any amount of pairs of zeros may be brought down as needed.

[72] For information on dealing with empirical amounts compare at the place sited page 224 and the following pages. **(Trans. Note: This appears to be an incomplete notation)**

```
√5 ≈ 2,23606
 4                    :  4₂ · 2
 10'0
   8 4
   1 60'0             :  44₃ · 3
   1 32 9
       27 1'0          :  446₆ · 6
       26 7 96
           3 040'00'0  :  4472₀₆ · 6
           2 683 23 6
             356 76 4
```

The approximate value we have found for √5 is √5 = 2.23606. It is:

2,23606^2 *≈4,99996,*

This means that the deviation first appears in the fifth decimal place. It is less than $\frac{1}{10000}$.

When calculating the fourth decimal place, a zero appears. In this case we have calculated further on the same line by "bringing down" two zeros. The same thing already happened once in the previous example. Therefore, there appear in the divisor *two* digits that are brought down. Of course, the zero will not be noted as a factor. Accordingly, multiple zeros also can successively appear.

Example 8:

Calculate √10 *reliably* up to the fifth decimal place.

```
√10  ≈  3,162277
  9
  10'0                    :  6₁ · 1
    6 1
    3 90'0                :  62₆ · 6
    3 75 6
      14 40'0              :  632₂ · 2
      12 64 4
        1 75 60'0         :  6324₂ · 2
        1 26 48 4
          49 11 60'0      :  63244₇ · 7
          44 27 12 9
            4 84 47 10'0     : 632454₇ · 7
            4 42 71 82 9
               41 75 27 1
```

The sixth decimal place is useful to round the fifth place up or down. The approximate value reliable to the fifth decimal place is:

$\sqrt{10} \approx 3{,}16228$

The answer check gives us:

3,16228^2 *≈ 10,00001489.*

The square of the approximate value is a good $\frac{1}{100000}$ too large.

At the same time, with √10, the approximate root values of all the odd exponents of 10 have been won. For example:

$$\sqrt{1000} = \sqrt{10^3} = \sqrt{10^2} \cdot \sqrt{10^1} = 10 \cdot \sqrt{10} \approx 31{,}6227$$

Or

$$\sqrt{100000} = \sqrt{10^5} = \sqrt{10^4} \cdot \sqrt{10^1} = 100 \cdot \sqrt{10} \approx 316{,}227.$$

But also, this applies to 0.1:

$$\sqrt{0{,}1} = \sqrt{\frac{10}{100}} = \frac{1}{10} \cdot \sqrt{10} \approx 0{,}316227$$

Or

$$\sqrt{0{,}001} = \sqrt{\frac{10}{10000}} = \frac{1}{10} \cdot \sqrt{10} \approx 0{,}0316227.$$

Basically, it is enough to find the root value between 1 and 100 because above and below, the succeeding digits are repeated. This is made use of with tables in that one gives the root value for a large enough amount of numbers between 1 and 100. These tables then allow one to look up the root value for any number with enough accuracy for practical purposes – just in case one does not want to use a calculator.

Finding Roots is very easy

If I will a root extract,
It’s oh so simple, and that’s a fact.
First I get the root all ready
By marking the number groups nice and steady.
During the process I must be deft,
Start from the decimal, go right and left.
And very soon I know without question,
For every group there is a position!
a is the first that we have named,
b is the second, and we’ve done the same.
The a I find as easy as pie,
by extracting the root in the blink of an eye.
But only from the very first group,
For the others, right now, I don’t give a hoot.
Then I subtract, most accurate and fair,
Not only the a, but the a as a square.
I supplement each time all the remainder,
By bringing on down the very next number.
To find the b, that is no trouble,
Just take the remainder and divide by a’s double.
When b is there for all to see,
I take away two a’s times b;
Follow then this advice most rare,
And be sure to subtract that little b square.
If a zero appears below the line,
I am so happy; I’ve done just fine.
And this I tell you right now and straight out,
I found the root quickly; of that there’s no doubt.
Harry Werner
Translated from German

by Nina Kuettel

Practice 61

1. Calculate $\sqrt{2}$, $\sqrt{5}$, $\sqrt{200}$, $\sqrt{1000}$, $\sqrt{2000}$, $\sqrt{4000}$, $\sqrt{7}$, $\sqrt{700}$ up to five decimal places each.

2. Calculate √10 from √2 and √5. Do the same for √14 from √2 and √7, as well as √35 from √5 and √7.

3. How large is the side of a square whose area is 9216cm^2?

4. A field shaped like a square measures 2ha **(Trans. Note: ha stands for hectares)** (1ha = 10,000 m^2). How large is the circumference? (Put *no* digit after the decimal point!)

5. How large is the side of a square that is the same size as two other squares together that have side lengths of 10cm and 20cm respectively?

6. How large is the hypotenuse of a right triangle whose legs are 208m and 228m? (Calculate up to dm exactly!)

7. The hypotenuse of a right triangle is 100m and one leg is 40m. How large is the other leg? (Calculate up to two decimal places!)

8. A bus drops off a group of hikers at the edge of some woods. The bus must drive around two streets that are at right angles to each other. The hikers walk on a straight path to their new destination. The bus must first go 2.5km on one street and then 3.7km on the other. How much more distance does the bus travel than the hikers?

9. A rectangular grass lawn, where no walking is allowed, has side lengths of 40m and 20m. But, some inattentive pedestrians take a short cut diagonally across the lawn. How many meters of walking distance have they spared?

10. Measure the sides of an A-4 size sheet of paper and calculate the length of the diagonal. Check the answer by re-measuring.

11. The bottom of a ladder is standing 2.5m away from a wall. How high is the ladder in order to reach the top of the wall that is 5m high?

12. How far from the wall of an 8m-high gable is a 10m-long beam if it reaches just to the tip of the gable?

13. A 12m-long ladder is set against the wall of a house so that the foot of the ladder is 2.5m distant from the wall. How far above the ground is the top end of the ladder?

14. A roof that is 6m high is put on a house that is 14m wide. What is the minimum length of the rafters?

15. What is the height of an equilateral triangle if one side is 10cm?

16. An object in a balance scale that is not quite exactly balanced weighs 47.5g on one scale pan and only 45.7g on the other pan. What is the true weight of the object? (Multiply both measured weights and find the square root of the product.)

Solutions:

1,4142; 2,2361; 14,142; 31,623; 44,721; 63,246; 2,6458; 26,458.

3,1623; 3,7417; 5,9163 (A more exact value by direct calculation is 5.9161.);

96cm; 4. 566m; 5. 22,361cm; 6. 308,6m; 7. 91,65m; 8. 1,73km; 9. 15,3m; 10. 36,37cm; 11. 5,59m; 12. 6m; 13, 11,74m; 14. 9,22m; 15. 8,66cm; 16. 46,6g.