I Multiplying and Dividing with Positive and Negative Numbers

I.1 Introduction

Up to now we have concentrated on adding and subtracting when dealing with negative numbers. How are negative numbers multiplied and divided? We will proceed step by step and begin with multiplying:

I.2 Multiplying Positive and Negative Numbers

1. We remember how we can get from adding to multiplying. For example, if we have the sum 3 + 4, multiplying is not considered. But if we have multiple sums with the same summands like 4 + 4 + 4 + 4 + 4, for example, we can count the summands and write that number in front of the multiplicand as the multiplier.

$$\underbrace{4+4+4+4+4}_{5} = 5 \cdot 4.$$

Correspondingly, we can also do this when the summands are negative numbers, like this, for example:

$$\underbrace{(-3)+(-3)+(-3)+(-3)+(-3)}_{5} = 5 \cdot (-3) = -15$$

Generally speaking, we can change a sum into a product if all the summands are the same:

$$\underbrace{a+a+...+a}_{n} = n \cdot a$$

The *multiplier*, the *counting* number, must of course be a natural number that is greater than 1. We may see that 1 · a = a and 0 · a = 0, but if we want to calculate (-3) · 4, then we must write out the 4 (-3) times. How should we do this? If we know that the commutative property also applies to negative numbers then we can assume that (-3) · 4 = 4 · (-3) = -12. However, in this world one cannot simply assume something but must check to see if that assumption is correct, as is the case here with the application of the commutative property.

At least we will try, through the use of these examples. We will choose a series of problems that show us that calculating -4 · 3 as 3 · (-4) easily fits into the calculations we have known up to now:

$3 \cdot 4 = 12$

$2 \cdot 4 = 8$

$1 \cdot 4 = 4$

$0 \cdot 4 = 0$

$-1 \cdot 4 = -4$

$-2 \cdot 4 = -8$

$-3 \cdot 4 = -12$ ....

Here we can see how each of the results are 4 less when the multiplier on the left decreases by 1 each time. We continue this in the logical manner when we transition from a positive to a negative multiplier with the result being 4 less each time.

Mathematicians have actually been able to show that in the arithmetic and algebra that we are learning here there are no contradictions when one applies the commutative property to the entire set of positive and negative numbers, including the zero.

If we multiply two positive numbers, then we always get a positive number, but if we multiply a positive number by a negative number, or the other way around, we always get a negative number. In short, we can say:

Plus times plus equals plus, minus times plus equals minus, and plus times minus equals minus.

The question remains: How can we multiply two negative numbers? The distributive property can provide us with a hint: The expression

$(7 - 3) \cdot (9 - 5)$

can be calculated in various ways. First, we can calculate each bracket and multiply the results.

$(7 - 3) \cdot (9 - 5) = 4 \cdot 4 = 16$

Now, we understand the subtractions as additions of negative numbers and multiply it out:

$(7 + (-3)) \cdot (9 + (-5)) = 7 \cdot 9 + 7 \cdot (-5) + (-3) \cdot 9 + (-3) \cdot (-5).$

According to the rule that we already know:

$7 \cdot 9 + 7 \cdot (-5) + (-3) \cdot 9 + (-3) \cdot (-5) = 63 - 35 - 27 + (-3) \cdot (-5) = 1 + (-3) \cdot (-5).$

So that we get the same result as the first calculation, this must be:

$(-3) \cdot (-5) = 15$

This result is to be interpreted exactly: Strictly speaking, it does *not* prove that minus · minus = plus. It says: When the distributive property is applied we must assume that minus · minus = plus. Using the new math it can be shown that it is possible to have algebraic forms that are not contradictory in which minus · minus = plus does *not* apply. However, one must refrain from using such a rule as the distributive property.

Again, a series of problems can show us how logically the rule minus · minus = plus fits into the structure of the calculations we have previously become familiar with:

$3 \cdot (-4) = -12$

$2 \cdot (-4) = -8$

$1 \cdot (-4) = -4$

$0 \cdot (-4) = 0$

$-1 \cdot (-4) = 4$

$-2 \cdot (-4) = 8$

$-3 \cdot (-4) = 12 \ldots$

By decreasing the multiplier by 1 every time, each result is decreased by 4. This continues into the transition to a multiplier that is a negative number. ,

The rule for multiplying positive and negative numbers can be summarized like this:

The product of two numbers is positive if their signs are the same, and negative when they are different.

The same thing expressed in formulas gives the following four cases:

1. $(+a) \cdot (+b) = +(a \cdot b),$
2. $(-a) \cdot (+b) = -(a \cdot b),$
3. $(+a) \cdot (-b) = -(a \cdot b),$
4. $(-a) \cdot (-b) = +(a \cdot b).$

There are several explanations for the multiplication rules that may help us remember, but are certainly *not* proofs. One example is: If one looks at love as positive and hate as negative, one can say: I love love. That is positive. I love hate. That is negative. I hate love. That is also negative. I hate hate. This, again, can be seen as positive when one thinks that hating every feeling of hatred would have the effect of erasing the feeling.

Also, spatial images, that are often used when introducing negative numbers, give no proof of this or other rules. They are images that show the application of algebraic structure to spatial relationships.

Note: If multiplication of positive and negative numbers is introduced purely arithmetically, then many examples should be given that strengthen the impression that the rules make sense.

First Example:

First of all, another table showing progressive numbers makes the rule more transparent. In the first box two positive numbers, including zero, are multiplied. In the next box to the right a positive number is multiplied by a negative number, and in the third box a negative by a negative.

	b = 3	2	1	0	-1	-2	-3
a = 3	*9*	*6*	*3*	*0*	*-1*	*-6*	*-9*
2	*6*	*4*	*2*	*0*	*-2*	*-4*	*-6*
1	*3*	*2*	*1*	*0*	*-1*	*-2*	*-3*
0	*0*	*0*	*0*	*0*	*0*	*0*	*0*
-1	*-3*	*-2*	*-1*	*0*	*1*	*2*	*3*
-2	*-6*	*-4*	*-2*	*0*	*2*	*4*	*6*
-3	*-9*	*-6*	*-3*	*0*	*3*	*6*	*9*

Second Example:

The area of a square with side length *a* is:

$$F = a^2$$

If I lengthen one side by c, and the other side by d, then naturally the whole area is changed. The new area F_1 is:

$$F_1 = (a + c)(a + d) = a^2 + ad + ac + cd. \; (*)$$

The change *A* in the area F is:

$$A = F1 - F = a^2 + ad + ac + cd - a^2 = ad + ac + cd. \; (**)$$

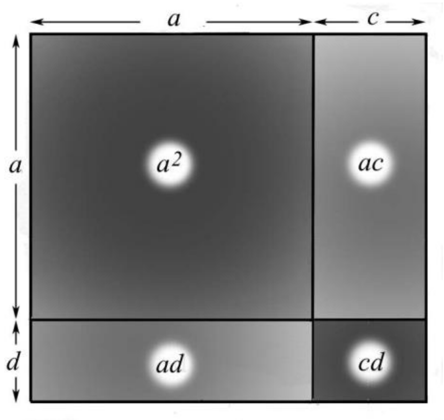


Fig. 1

For example, if a = 10cm, c = 3cm, and d = 2cm, then the change is:

$A = (10 + 3) \cdot (10 + 2)\ cm^2 - 100\ cm^2 = 100\ cm^2 + 20\ cm^2 + 30\ cm^2 - 100\ cm^2 = 56\ cm^2$

One can follow in the graphic how every partial product of an area adds to $a^{2.}$

Does the formula also apply if *c* or *d* or both are a negative number and there is a shortening instead of a lengthening? How do we calculate the product of two negative numbers so that the original formula is also applicable in this case? For example:

A = 10cm, c = -3cm, and d = -2cm.

The square will be made smaller. If we put these sizes into the area formula (*) for the changed square we get:

$F_1 = (a + c)(a + d) = [10 + (-3)][10 + (-2)] = (10 – 3)(10 – 2)$

One way of calculating it is this:

$F_1 = [10 + (-3)][10 + (-2)] = (10-3)(10-2) = 7 \cdot 8 = 56$

Another way is to multiply out the brackets [10 + (-3)][10 + (-2)] and, under the condition that *plus ·minus = minus · plus = minus,* we get:

$F_1 = [10 + (-3)][10 + (-2)] = 10 \cdot 10 + 10 \cdot (-2) + 10 \cdot (-3) + (-3) \cdot (-2) = 100 – 20 – 30 + (-3) \cdot (-2) = 50 + (-3) \cdot (-2).$

So that the single correct answer is found, one must reckon with *minus · minus = plus.*

According to (**) for the change we get:

$A = F_1 - F = 56cm^2 - 100cm^2 = -44cm^2$

This means that the area has gotten smaller. In the second graphic we have represented the previous case with two side length reductions.

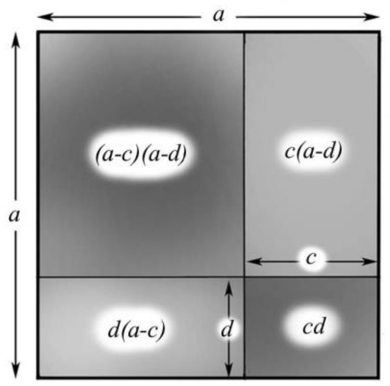


Fig. 2

Accordingly, the cases with reduction on only one side can be geometrically represented.

However, be aware that this is no proof of *minus · minus = plus*. It only shows that the rule makes sense for many different applications.

I.3 Dividing Positive and Negative Numbers

We know that division and multiplication are inversions of each other. That means, if we multiply a number by another number and then divide it with this number, we get the starting number again. The same thing happens if we first divide and then multiply with the same number. Multiplying and dividing with the same number cancels each other out. Examples of this are:

$(2 \cdot 3) : 3 = 2$ or generally for positive numbers a and b

$(a \cdot b): b = a$ written as a fraction $\frac{a \cdot b}{b} = a$ *or*

$(a : b) \cdot b = a$ written as a fraction $\left(\frac{a}{b}\right) \cdot b = a$.

Now, let us assume that the same rules apply to division with negative numbers. For an expression such as $2 \cdot (-3) = -6$, for example, through division by (-3) on both sides, we get

$2 = (-6) : (-3).$

Division of *two negative* numbers by each other results in a *positive* number. From the rules of multiplication already given, we can deduce the general rules for division in the area of whole numbers (with the exception of 0 as a divisor):

1. $(+a) : (+b) = +(a : b)$ beziehungsweise $\frac{+a}{+b} = +\frac{a}{b}$,
2. $(-a) : (+b) = -(a : b)$ beziehungsweise $\frac{-a}{+b} = -\frac{a}{b}$,
3. $(+a) : (-b) = -(a : b)$ beziehungsweise $\frac{+a}{-b} = -\frac{a}{b}$,
4. $(-a) : (-b) = +(a : b)$ beziehungsweise $\frac{-a}{-b} = +\frac{a}{b}$.

It does not matter if a or b is positive or negative.

Rule: *If the dividend and the divisor have the same sign then the result will be positive, but if the signs of the dividend and divisor are different, the result is negative.*

The basic principles of algebra in the area of whole numbers should be well practiced. Just as in music, one needs to build basic skills in algebra. If not, successful advancement in the subject is hard to imagine.

Practice 25

Calculate the results in two ways: 1. First multiply out the brackets and then add or subtract. 2. To check your answers, first calculate the brackets and then multiply them out.

1 a-d)	$(3 - 1)(3 + 1)$	$(4 + 3)(4 - 3)$	$(5 - 2)(5 + 2)$	$(7 + 6)(7 - 6)$
2 a-d)	$(3 - 1)(3 - 1)$	$(4 + 3)(4 + 3)$	$(5 - 2)(5 - 2)$	$(7 + 6)(7 + 6)$
3 a-d)	$(20 - 1)(20 + 1)$	$(20 - 2)(20 + 2)$	$(20 - 3)(20 + 3)$	$(20 - 4)(20 + 4)$
4 a-d)	$(20 + 1)(20 + 1)$	$(20 + 2)(20 + 2)$	$(20 + 3)(20 + 3)$	$(20 + 4)(20 + 4)$
5 a-d)	$(20 - 1)(20 - 1)$	$(20 - 2)(20 - 2)$	$(20 - 3)(20 - 3)$	$(20 - 4)(20 - 4)$
6 a-d)	$(14 + 6)(23 - 3)$	$(22 - 2)(27 - 7)$	$(31 - 1)(27 + 3)$	$(99 + 1)(101 - 1)$

Solutions:

1a) 8; b) 7; c) 21; d) 13; 2a) 4; b) 49; c) 9; d) 169; 3a) 399; b) 396; c) 391; d) 384; 4a) 441; b) 484; c) 529; d) 576; 5a) 361; b) 324; c) 289; d) 256; 6a) 400; b) 400; c) 900; d) 10.000.

Practice 26

Calculate the following products. Use $a \cdot a = a^2$ and the same for other letters:

1 a-d)	$(a + 1)(a + 2)$	$(b + 2)(b +3)$	$(c + 3)(c + 4)$	$(d + 4)(d + 5)$
2 a-d)	$(a - 1)(a + 2)$	$(a + 1)(a - 2)$	$(a - 1)(a - 2)$	$(a - 6)(a + 7)$
3 a-d)	$(2a +3)(2a + 3)$	$(2a +3)(2a - 3)$	$(2a - 3)(2a - 3)$	$(-2a +3)(-2a + 3)$
4 a-d)	$(-2a +3)(2a + 3)$	$(-2a +3)(-2a - 3)$	$(-2a - 3)(-2a - 3)$	$(-2a - 3)(2a + 3)$
5 a-d)	$\left(\frac{v}{a}-\frac{w}{a}\right)\left(\frac{w}{a}+\frac{v}{a}\right)$	$\left(\frac{d}{2}+\frac{1}{3}\right)\left(\frac{d}{2}+\frac{1}{3}\right)$	$\left(\frac{d}{2}-\frac{1}{3}\right)\left(\frac{d}{2}+\frac{1}{3}\right)$	$\left(\frac{d}{2}-\frac{1}{3}\right)\left(\frac{d}{2}-\frac{1}{3}\right)$

Solutions:

1a) $a^2 + 3a + 2$*; b)* $b^2 + 5b + 6$; *c)* $c^2 + 7c + 12$*; d)* $d^2 + 9d + 20$*; 2a)* $a^2 + a - 2$*; b)* $a^2 - a - 2$*; c)* $a^2 - 3a + 2$*; d)* $a^2 + a - 42$*; 3a)* $4a^2 + 12a + 9$*; b)* $4a^2 - 9$*; c)* $4a^2 - 12a + 9$*; d)* $4a^2 -12a + 9$*; 4a)* $-4a^2 + 9 = 9 - 4a^2$*; b)* $4a^2 - 9$*; c)* $4a^2 + 12a + 9$*; d)* $-4a^2 - 12a - 9$*; 5a)* $\frac{v^2}{a^2}-\frac{w^2}{a^2}$*; b)* $\frac{d^2}{4}+\frac{d}{3}+\frac{1}{9}$*; c)* $\frac{d^2}{4}-\frac{1}{9}$*;*

d) $\frac{d^2}{4}-\frac{d}{3}+\frac{1}{9}$

Practice 27

Go step by step and remove all the brackets and then calculate the result:

1. *7 – [4 – 8 – (3 – 4)]*
2. *-9 – [8 – 7 – (6 + 5)]*
3. *11 – [-12 – 1 – (2 – 14)]*
4. *9 – [13 – 4·(19 – 20 + 2)]*
5. *17 – [18 – 5·(16 – 13)]*
6. *-111 + 100·[25 – 3·(12 – 4)]*
7. *4·[9 – 4·(9 – 7)] – 6·[4 – 3·(9 – 3 – 5)]*
8. *12·[17 – 4·(26 – 22)] + 13·[34 – 5·(33 – 26)]*
9. *(9-12)·{77 – {66 – [55 – (44 – 33)]}*
10. *(11 – 7)·{84 – [72 – (60 – 48)]}*

Solutions:

1. *10*; 2. *1*; 3. *12*; 4. *0*; 5. *14*; 6. *-11*; 7. *-2*; 8. *-1*; 9. *-165* ; 10. *96*.

Practice 28

Calculate:

1 a-d)	$12 : 4$	$12 : (-4)$	$-12 : 4$	$-12 : (-4)$
2 a-d)	$121a : 11ab$	$-11ab : 22b$	$33ab : (-11a)$	$1001rs : 11r$
3 a-d)	$\frac{13abc-39ab}{13a}$	$\frac{14xyz-42xy}{14xy}$	$\frac{51rst-68rt}{17rt}$	$\frac{1001uv-143v}{143v}$

4 a-d) $(26rs - 52s) : 13s$ $(18xy - 10yz) : 4y$ $(21k - 33kn) : 9k$ $(52ijk + 78jk) : 39jk$

Solutions:

1a) 3; b) –3; c) –3; d) 3; *2a)* $\frac{11}{b}$ *; b)* $-\frac{1}{2}a$ *; c) –3b; d) 91s;*

3a) bc – 3b = b · (c – 3); b) z – 3; c) 3s - 4; 3d) 7u – 1;

4a) 2r – 4; b) $\frac{9}{2}x - \frac{5}{2}z = \frac{9x-5z}{2}$ *; c)* $\frac{7}{3} - \frac{11}{3}n = \frac{7-11n}{3}$ *; d)* $\frac{4}{3}i + 2$.

Practice 29

From time to time mixed numbers should be brought in, even if they have no great mathematical significance. The problem is that a number like $6\frac{2}{5}$ actually should be written $6 + \frac{2}{5}$. Because of the missing + one could think of a product. The advantage of mixed numbers is that it is easy to estimate their size: $6\frac{2}{5}$ is easier to estimate than $\frac{32}{5}$.

1. *Example problem:* There are different ways to calculate $3\frac{7}{10} \cdot 5$:

a) If it is easily done, change the mixed number into a decimal number and then multiply:

$3\frac{7}{10} \cdot 5$ = 3.7 · 5 = 18.5.

b) Write the mixed number as a sum and then multiply it:

$3\frac{7}{10} \cdot 5 = (3 + \frac{7}{10}) \cdot 5 = 15 + \frac{7}{2} = 18\frac{1}{2}$

c) Change the mixed numbers into so-called improper fractions and multiply:

$3\frac{7}{10} \cdot 5 = \frac{37}{10} \cdot 5 = \frac{37}{2} = 18\frac{1}{2}$

2. *Example problem:* Calculate $3\frac{7}{10} \cdot 2\frac{1}{4}$ in different ways.

a) $3\frac{7}{10} \cdot 2\frac{1}{4} = 3{,}7 \cdot 2{,}25 = 8{,}325$

b) $3\frac{7}{10} \cdot 2\frac{1}{4} = (3 + \frac{7}{10})(2 + \frac{1}{4}) = 6 + \frac{3}{4} + \frac{7}{5} + \frac{7}{40} = \frac{333}{40} = 8\frac{13}{40}$

c) $3\frac{7}{10} \cdot 2\frac{1}{4} = \frac{37}{10} \cdot \frac{9}{4} = \frac{333}{40} = 8\frac{13}{40}$

Practice 30

Calculate the following products in different ways:

1. $3\frac{3}{5} \cdot 7\frac{3}{10}$; 2. $1\frac{1}{4} \cdot 2\frac{1}{5}$; 3. $1\frac{1}{3} \cdot 2\frac{2}{3}$; 4. $12\frac{4}{7} \cdot 7\frac{1}{4}$; 5. $15\frac{5}{9} \cdot 18\frac{3}{5}$

Solutions:

1. $26\frac{7}{25} = 26{,}28$; 2. $2\frac{3}{4} = 2{,}75$; 3. $3\frac{5}{9}$; 4. $91\frac{1}{7}$; 5. $289\frac{1}{3}$

I.4 The Binomial Formulas

We have been occupied with multiplication of sums. A few important exceptions are found in the known binomial formulas that we can easily gain by multiplying out:

I. Binomische Formel

$(a + b)^2 = (a + b)\cdot(a + b) = a \cdot a + a \cdot b + b \cdot a + b \cdot b = a^2 + 2ab + b^2$

II. Binomische Formel

$(a - b)^2 = (a - b)\cdot(a - b) = a^2 - ab - ba + b^2 = a^2 - 2ab + b^2$

III. Binomische Formel

$(a + b)\cdot(a - b) = a^2 - ab + ba - b^2 = a^2 - b^2$

Even though the main significance of these formulas does not lie in their use for numbers, still, they can be advantageously used for calculating some products.

Practice 31

1. Calculate 63^2.

Solution: We use formula I and put in a = 60, b = 3. We then have

$63^2 = (60 + 3)^2 = 60^2 + 2 \cdot 60 \cdot 3 + 3^2 = 3600 + 360 + 9 = 3969$

2. Calculate 69^2.

Solution: We use formula II and put in a = 70, b = 1. We then have

$69^2 = (70 - 1)^2 = 70^2 - 2 \cdot 70 \cdot 1 + 1^2 = 4900 – 140 + 1 = 4761.$

3. Calculate 68 · 72.

Solution: We use formula III and put in a = 70, b = 2. We then have

$68 \cdot 72 = (70 + 2)(70 – 2) = 70^2 – 2^2 = 4900 – 4 = 4896.$

4. Calculate using the binomial formulas

a) 38 · 42; b) 38^2; c) 42^2; d) 67^2; e) 73^2; f) 67 · 73; g) 101^2; h) 99^2; i) 99 · 101.

Solutions: a) *1596*; b) *1444*; c) *1764*; d) *4489*; e) *5329*; f) *4891*; g) *10201*; h) *9801*; i) *9999*.

The binomial formulas can be expanded in two directions: We can increase the number of summands in the brackets and we can calculate higher than the second power. Both expansions play a role in mathematics. Let us begin with the first: If we multiply the sum (a + b + c) by itself, then after ordering and combining, we get:

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc = a^2 + b^2 + c^2 + 2\cdot(ab + ac + bc)$

Practice 32

1. With the help of the previous formula calculate 243^2.

Solution:

$$
\begin{array}{rr}
243^2 = (200+40+3)^2 = & 40000 \\
& 1600 \\
& 9 \\
& 16000 \\
& 1200 \\
& 240 \\
\hline
& \underline{\underline{59049}}
\end{array}
$$

2. Calculate 125^2, 128^2, 289^2.

Solutions: 15625; 16384; 83521.

3. The previous examples also show the following: The last digit of the result of squaring a multi-digit number depends only upon the last digit of this number. All other summands are multiples of 10. Later, this may help with quickly finding whole number roots.

4. Use a formula to change the expression $(2a + 3b - c)^2$ into a sum.

Solution: : $4a^2 + 9b^2 + c^2 + 12ab - 4ac - 6bc$.

The second expansion mentioned above leads to higher powers of a binomial. We calculate:

$$(a + b)^3 = (a + b)^2 \cdot (a + b) = (a^2 + 2ab + b^2)(a + b) = a^3 + 3a^2b + 3ab^2 + b^3$$

With this we could calculate 43^3, for example:

$$\begin{array}{rr} 43^3 = (40+3)^3 = & 64000 \\ & 14400 \\ & 1080 \\ & 27 \\ \hline & \underline{\underline{79507}} \end{array}$$

Practice 33

Calculate using the previous formula

11^3; 17^3; 32^3; 55^3; 99^3; $(2a + b)^3$; $(0{,}5r + 2s)^3$.

Solutions: 1331; 4913; 32768; 166375; 970299; $8a^3 + 12a^2b + 6ab^2 + b^3$;

$0{,}125r^3 + 1{,}5r^2s + 6rs^2 + 8s^3$.

I.5 An Important Practice Series

We now turn to one of the most important practice groups that must be very carefully handled. It has to do with multiplying out brackets and reconverting appropriate sums into products. We have discussed and thoroughly practiced how sums can be multiplied together. We have also practiced raising sums into products by extracting the common factors of all the summands. The basis for this was the distributive and commutative properties. We will now discuss the important case in which a sum is changed into a product of sums; each factor of the product is again a sum.

Let us begin with a simple example. The product

$$(a + 3)\cdot(a + 4)$$

is a product of two sums. When it is multiplied out we get:

$$(a + 3)\cdot(a + 4) = a^2 + 3a + 4a + 12 = a^2 + 7a + 12.$$

Because a appears in both brackets we can combine the result into a three-part sum.

We ask: If $a^2 + 7a + 12$ is before us, how can we get back to the original product? We must be clear about how every part of the sum came about. The a^2 comes from $a \cdot a$. We get this when we begin both brackets with a:

$$(a \pm \ldots)(a \pm \ldots).$$

But this would also be possible:

$(-a \pm \ldots)(-a \pm \ldots)$, weil ja $(-a)(-a) = +a^2$ ist.

The term 7a is the sum of 3a + 4a. 7 is the sum of the two numbers in the brackets. 7 can be created as a sum in many different ways. If we limit ourselves to whole numbers, then

$7 = 7 + 0 = 6 + 1 = 5 + 2 = \ldots$ or $7 = 8 + (-1)$ *etc.*

Since we know that the 7 in 7a is the sum of both number terms in the brackets, it is not yet to be determined. We must take into consideration that 12 is the *product* of these two numbers. However, this alone would not help much because 12 can be expressed as a product in many ways. For example:

$12 = 1 \cdot 12 = 2 \cdot 6 = 3 \cdot 4 = \frac{1}{2} \cdot 24$, *etc.*

Only *both* conditions together are of further help to us: Which two numbers have both the sum of 7 and the product of 12? These are in fact only the numbers 3 and 4. It is then:

$a^2 + 7a + 12 = (a + 3)(a + 4).$

With that, the re-conversion is successful.

Obviously, this was a very “accommodating” example. How should we have proceeded if the given sum was $a^2 + 7a + 13$? One sees that multiplying out the brackets, meaning multiplying the sums, does not require great skill. However, raising a sum into a product is often a difficult task and needs to be practiced. So, let us do a series of problems right away.

Practice 34

Change the following sums into products of two sums, and, to check your answers, multiply out the brackets:

1.	$a^2 + 5a + 6$	8.	$a^2 + 7a + 12$
2.	$a^2 + 5a + 4$	9.	$a^2 + 8a + 7$
3.	$a^2 + 6a + 5$	10.	$a^2 + 8a + 12$
4.	$a^2 + 6a + 8$	11.	$a^2 + 8a + 15$
5.	$a^2 + 6a + 9$	12.	$a^2 + 8a + 16$
6.	$a^2 + 7a + 6$	13.	$a^2 + 10a + 25$
7.	$a^2 + 7a + 10$	14.	$a^2 + 12a + 36$

15. Think of sums that can be changed into products with whole numbers and let your neighbor solve them.

Solutions:

1. $(a + 2)(a + 3)$; 2. $(a + 1)(a + 4)$; 3. $(a + 1)(a + 5)$; 4. $(a + 2)(a + 4)$; 5. $(a + 3)(a + 3)$; 6. $(a + 1)(a + 6)$; 7. $(a + 2)(a + 5)$; 8. $(a + 3)(a + 4)$; 9. $(a + 1)(a + 7)$; 10. $(a + 2)(a + 6)$; 11. $(a + 3)(a + 5)$; 12. $(a + 4)(a + 4)$; 13. $(a + 5)(a + 5)$; 14. $(a + 6)(a + 6)$.

In the next group of problems we add fractions. How can one change the following into a product?

$$a^2 + \frac{9}{2}a + 2$$

Obviously, the fraction with the denominator 2 plays a role. However, the second number must be whole, or, at the end, instead of 2, there would be a fraction. Let us try the simplest case: $4 \cdot \frac{1}{2} = 2$.

$\frac{9}{2}$xxx

At the same time the sum of the factors must give the number for a. This is in fact the case,

because $4 + \frac{1}{2} = \frac{9}{2}$. So, it is:

$$a^2+\tfrac{9}{2}a+2=(a+\tfrac{1}{2})(a+4)$$

Practice 35

Change the following sums into products of two sums and check your answers by multiplying out the brackets:

1.	$a^2+\frac{5}{2}a+1$	8.	$a^2+\frac{4}{3}a+\frac{1}{3}$
2.	$a^2+\frac{9}{2}a+2$	9.	$a^2+\frac{2}{3}a+\frac{1}{9}$
3.	$a^2+\frac{13}{2}a+3$	10.	$a^2+\frac{4}{9}a+\frac{1}{27}$
4.	$a^2+\frac{17}{2}a+4$	11.	$a^2+\frac{3}{4}a+\frac{1}{8}$
5.	$a^2+\frac{10}{3}a+1$	12.	$a^2+\frac{7}{12}a+\frac{1}{12}$
6.	$a^2+\frac{19}{3}+2$	13.	$a^2+\frac{1}{2}a+\frac{1}{16}$
7.	$a^2+\frac{28}{3}a+3$	14.	$a^2+\frac{13}{12}a+\frac{5}{24}$

Solutions:

1. $(a+\frac{1}{2})(a+2)$; 2. $(a+\frac{1}{2})(a+4)$; 3. $(a+\frac{1}{2})(a+6)$; 4. $(a+\frac{1}{2})(a+8)$; 5. $(a+\frac{1}{3})(a+3)$; 6. $(a+\frac{1}{3})(a+6)$; 7. $(a+\frac{1}{3})(a+9)$; 8. $(a+\frac{1}{3})(a+1)$; 9. $(a+\frac{1}{3})(a+\frac{1}{3})$; 10. $(a+\frac{1}{3})(a+\frac{1}{9})$; 11. $(a+\frac{1}{4})(a+\frac{1}{2})$; 12. $(a+\frac{1}{4})(a+\frac{1}{3})$; 13. $(a+\frac{1}{4})(a+\frac{1}{4})$; 14. $(a+\frac{1}{4})(a+\frac{5}{6})$.

Naturally, instead of sums there could also be differences. If we multiply the two differences

(a – 2)(a – 3) together we must pay close attention to the rule about prefix signs. It is:

$$(a-2)(a-3) = a^2-5a+6.$$

The prefix signs in the answer show that positive as well as negative prefix signs must appear in brackets. Let us check which cases are actually possible and which ones lead to our result. We will think of any two brackets with two-digit sums:

$(a+b)(c+d) = ac+ad+bc+bd$	$(-a-b)(-c-d) = ac+ad+bc+bd$
$(a+b)(c-d) = ac-ad+bc-bd$	$(-a-b)(-c+d) = ac-ad+bc-bd$
$(a-b)(c+d) = ac+ad-bc-bd$	$(-a+b)(-c-d) = ac+ad-bc-bd$
$(a-b)(c-d) = ac-ad-bc+bd$	$(-a-b)(-c-d) = ac-ad-bc+bd$
$(-a+b)(c+d) = -ac-ad+bc+bd$	$(a+b)(-c+d) = -ac-ad+bc+bd$
$(-a+b)(c-d) = -ac+ad+bc-bd$	$(a+b)(-c-d) = -ac+ad+bc-bd$
$(-a-b)(c+d) = -ac-ad-bc-bd$	$(a-b)(-c+d) = -ac-ad-bc-bd$
$(-a-b)(c-d) = -ac+ad-bc+bd$	$(a-b)(-c-d) = -ac+ad-bc+bd$

We can see that essentially eight different cases occur. The ones that are next to each other lead to the same result because the brackets are equal if we factor out the *-1* from them. With $(-1)\cdot(-1) = 1$ the result is unchanged.

If we change a sum into a product, it is only possible up to the opposite prefix signs of all the summands. However, in the following exercises we will content ourselves with only one solution, knowing that by reversing all the prefix signs a second solution is given.

If we choose a more systematic approach to changing sums into products we will have to pay close attention to the prefix signs.

In

$$a^2+5a+6$$

all the prefix signs are positive; we must find two numbers whose sum is 5 and whose product is 6. 2 and 3 fulfill these conditions. The sum

$a^2 - a - 6$

is already more difficult. The a^2 shows us that the form of the product will be

(a…)(a…).

Since the pure number digit is negative, both the numbers in the brackets must have the opposite prefix signs. However, because the prefix sign of *a* is negative (-1), the larger of the two numbers must be negative.

If we write

$(a + 2)(a - 3),$

when we multiply it out, we get the sum:

$(a + 2)(a - 3) = a^2 - a - 6.$

Raising the sum to a product is successful. Of course, one cannot generally assume that such transformations with whole numbers will always work. Sufficient practice is the only way to achieve the desired skill level.

Practice 36

Change the following three-element expressions into a product of two sums or differences:

1.	$a^2 + 5a + 6$	9.	$x^2 + 7x + 10$
2.	$a^2 - a - 6$	10.	$x^2 + 3x - 10$
3.	$a^2 + a - 6$	11.	$x^2 - 3x - 10$
4.	$a^2 - 5a + 6$	12.	$x^2 - 7x + 10$
5.	$-a^2 - a + 6$	13.	$-x^2 + 3x + 10$
6.	$-a^2 + 5a - 6$	14.	$-x^2 + 7x - 10$
7.	$-a^2 - 5a - 6$	15.	$-x^2 - 7x - 10$
8.	$-a^2 + a + 6$	16.	$-x^2 - 3x + 10$

Solutions:

1. $(a + 2)(a + 3)$; 2. $(a + 2)(a - 3)$; 3. $(a - 2)(a + 3)$; 4. $(a - 2)(a - 3)$;5. $(-a + 2)(a + 3)$; 6. $(-a + 2)(a - 3)$; 7. $(-a - 2)(a + 3)$; 8. $(-a - 2)(a - 3)$; 9. $(x + 2)(x + 5)$; 10. $(x - 2)(x + 5)$; 11. (x + 2)(x –5); 12. $(x - 2)(x - 5)$; 13. $(-x - 2)(x - 5)$; 14. $(-x + 2)(x - 5)$; 15. $(-x - 2)(x + 5)$; 16 $(-x + 2)(x + 5)$.

The conversions that we are practicing do not depend upon the letters used. For some students it is important to emphasize from time to time that letters cannot be added together or multiplied. They simply serve as *place holders* for numbers, as has already been explained. Where there is a letter, any number can be substituted, as long as the same number always stands for the same letter in an equation. The conversions show that for any numbers, different orders of mathematical operations all lead to the same result. It may be helpful to practice sometimes putting numbers in place of letters.

Practice 37

1.	$a^2 + 3a + 2$	13.	$n^2 - 2n - 15$
2.	$b^2 + 5b + 6$	14.	$p^2 + 3p - 4$
3.	$c^2 + 7c + 12$	15.	$q^2 - 3q - 4$
4.	$d^2 + 9d + 20$	16.	$r^2 + 3r - 10$
5.	$e^2 + 11e + 30$	17.	$s^2 - 3s - 10$
6.	$f^2 + 13f + 42$	18.	$t^2 - 7t + 10$
7.	$g^2 + 15g + 56$	19.	$u^2 + 9u + 18$
8.	$h^2 + 2h - 3$	20.	$v^2 + 3v - 18$

9.	$i^2 - 2i - 3$	21.	$w^2 - 3w - 18$
10.	$j^2 + 2j - 8$	22.	$x^2 - 9x + 18$
11.	$k^2 - 2k - 8$	23.	$y^2 - 6y + 5$
12.	$m^2 + 2m - 15$	24.	$z^2 - 8z + 12$

Solutions:

1. $(a + 1)(a + 2)$; 2. $(b + 2)(b + 3)$; 3. $(c + 3)(c + 4)$; 4. $(d + 4)(d + 5)$;5. $(e + 5)(e + 6)$; 6. $(f +6)(f + 7)$; 7. $(g + 7)(g + 8)$; 8. $(h - 1)(h + 3)$; 9. $(i + 1)(i - 3)$; 10. $(j - 2)(j + 4)$; 11. $(k + 2)(k - 4)$; 12. $(m - 3)(m + 5)$; 13. $(n + 3)(n - 5)$; 14. $(p - 1)(p + 4)$; 15. $(q + 1)(q - 4)$; 16. $(r - 2)(r + 5)$; 17. $(s + 2)(s - 5)$; 18. $(t - 2)(t - 5)$; 19. $(u + 3)(u + 6)$; 20. $(v - 3)(v +6)$; 21. $(w + 3)(w - 6)$; 22. $(x - 3)(x - 6)$; 23. $(y - 1)(y -5)$; 24. $(z - 2)(z - 6)$.

The problem exercises that lead to the binomial formulas are especially important:

Practice 38

1.	$a^2 + 2a + 1$	9.	$x^2 - 1$
2.	$a^2 -2 a + 1$	10.	$x^2 - 4$
3.	$a^2 + 4a + 4$	11.	$x^2 - 9$
4.	$a^2 - 4a + 4$	12.	$x^2 - 100$
5.	$a^2 + 6a + 9$	13.	$-x^2 + 121$
6.	$a^2 - 6a + 9$	14.	$x^2 - 361$
7.	$a^2 + 8a + 16$	15.	$-x^2 + 289$
8.	$a^2 - 8a + 16$	16.	$x^2 - 256$

Solutions:

1. $(a + 1)^2$; 2. $(a - 1)^2$; 3. $(a + 2)^2$; 4. $(a - 2)^2$;5. $(a + 3)^2$; 6. $(a - 3)^2$; 7. $(a + 4)^2$; 8. $(a - 4)^2$; 9. $(x + 1)(x - 1)$; 10. $(x + 2)(x - 2)$; 11. $(x + 3)(x - 3)$; 12. $(x + 10)(x - 10)$; 13. $(x + 11)(-x + 11)$; 14. $(x + 19)(x - 19)$; 15. $(x + 17)(-x + 17)$; 16. $(x + 16)(x - 16)$.

Converting sums with more than one place holder into products requires increased concentration. Doing a few problems of multiplying out, using the associated rules, can help one become familiar with them.

Practice 39

Multiply out and combine as far as possible:

1. (a + 3b)(a + 4b).
2. $(2a + 3b)(a + 4b)$.
3. $(2a + 3b)(4a + 5b)$.
4. (a + 3b)(a – 4b).
5. $(2a - 3b)(a + 4b)$.
6. $(2a - 3b)(4a - 5b)$.

07. (5a – 7b)(5a + 7b).

Solutions:

1. $(a + 3b)(a + 4b) = a^2 + 4ab + 3ab + 12b^2 = a^2 + 7ab + 12b^2$.
2. $(2a + 3b)(a + 4b) = 2a^2 + 8ab + 3ab + 12b^2 = 2a^2 + 11ab + 12b^2$.
3. $(2a + 3b)(4a + 5b) = 8a^2 + 10ab + 12ab + 15b^2 = 8a^2 + 22ab + 15b^2$.
4. $(a + 3b)(a - 4b) = a^2 - 4ab + 3ab + 12b^2 = a^2 - ab - 12b^2$.
5. $(2a - 3b)(a + 4b) = 2a^2 + 8ab - 3ab - 12b^2 = 2a^2 + 5ab - 12b^2$.

6. $(2a - 3b)(4a - 5b) = 8a^2 - 10ab - 12ab + 15b^2 = 8a^2 - 22ab + 15b^2$.
7. $(5a - 7b)(5a + 7b) = 25a^2 + 35ab - 35ab - 49b^2 = 25a^2 - 49b^2$.

Notes:

1. One possible source of failure with these problems is the following: If a sum (a + b) is to be multiplied with a number *c*, one must multiply each summand in the bracket with *c*. If (ab) is to be multiplied with *c*, some students will multiply both factors with *c*, like so: $c \cdot (ab) = ca \cdot cb$. This is not correct.

2. One must always pay attention that the operations are done in the following order as far as possible: Number – letters in alphabetical order, combining the same factors into powers.

By the use of some examples, we will now thoroughly discuss converting more complex sums into products.

Example Problems for Converting Complex Expressions into Products

Convert the following sums into products:

1. $a^2 + 10ab + 24b^2$

Solution: First, look at the prefix signs. Since only positive operation signs are present (+) we put a + sign in the brackets. Then we look at the square numbers. They show that the summands in the brackets must have this form:

$(a + \ldots b)(a + \ldots b)$.

The numbers in front of the *b* digits are still unknown. We will call them the prefix numbers of *b*. Just as in earlier exercises, their product must be 24 and their sum 10. The numbers 4 and 6 fulfill these conditions.

$(a + 4b)(a + 6b)$

Multiplying out the above will again lead to the given sum.

2. $a^2 - 4ab - 32b^2$

Solution: The two minus signs indicate that in one bracket the *b* digit must have a negative sign. Appropriate prefix numbers are 4 and -8. The negative sign must be in front of the number with the larger value so that the sum of the two prefix numbers will be negative. We have:

$(a + 4b)(a - 8b)$

Multiplied out we get the correct sum.

3. $25a^2 - 110ab + 121b^2$

Solution: Both of the square numbers are positive and the *mixed number* (we call it that because it contains both place holders) is negative. So, there must be a minus sign in both brackets. Since $25a^2 = (5a)^2$ and $121b^2 = (11b)^2$, we write:

$(5a - 11b)(5a - 11b)$

Now, above all, we must check that we get the correct answer when multiplying out the mixed number. That is the case here, and we have found the correct answer.

Practice 40

Convert the following sums into products of two-digit sums or differences. Always begin by looking at the prefix signs, then the squares, and finally the mixed numbers.

1.	$a^2 + 2ab + b^2$	13.	$4x^2 - 9y^2$

2.	$a^2 + 3ab + 2b^2$	14.	$4x^2 + 2xy - 30y^2$
3.	$a^2 + 4ab + 4b^2$	15.	$4x^2 - 25y^2$
4.	$a^2 + 4ab + 3b^2$	16.	$u^2 - 2uv + v^2$
5.	$a^2 + 5ab + 6b^2$	17.	$2u^2 - 3uv + v^2$
6.	$a^2 + 6ab + 9b^2$	18.	$2u^2 - 5uv + 2v^2$
7.	$a^2 + 5ab + 4b^2$	19.	$6u^2 - 13uv + 6v^2$
8.	$4x^2 - 2xy - 2y^2$	20.	$55u^2 - 146uv + 55v^2$
9.	$4x^2 - y^2$	21.	$91u^2 - 218uv + 91v^2$
10.	$4x^2 - 2xy - 6y^2$	22.	$2u^2 + \frac{63}{4}uv - 2v^2$
11.	$4x^2 - 4y^2$	23.	$2u^2 - \frac{63}{4}uv - 2v^2$
12.	$4x^2 + 2xy - 20y^2$	24.	$2u^2 + \frac{65}{4}uv + 2v^2$

Solutions:

1. $(a + b)(a + b) = (a + b)^2$; 2. $(a + 2b)(a + b)$; 3. $(a + 2b)(a + 2b) = (a + 2b)^2$; 4. $(a + 3b)(a + b)$; 5. $(a + 3b)(a + 2b)$; 6. $(a + 3b)(a + 3b) = (a + 3b)^2$; 7. $(a + 4b)(a + b)$; 8. $(2x - 2y)(2x + y)$; 9. $(2x - y)(2x + y)$; 10. $(2x - 2y)(2x + 3y)$; 11. $(2x - 2y)(2x + 2y)$; 12. $(2x - 4y)(2x + 5y)$; 13. $(2x - 3y)(2x + 3y)$; 14. $(2x - 5y)(2x + 6y)$; 15. $(2x - 5y)(2x + 5y)$; 16. $(u - v)(u - v) = (u - v)^2$; 17. $(2u - v)(u - v)$; 18. $(2u - v)(u - 2v)$; 19. $(2u - 3v)(3u - 2v)$; 20. $(5u - 11v)(11u - 5v)$; 21. $(7u - 13v)(13u - 7v)$; 22. $(4u-\frac{1}{2}v)(\frac{1}{2}u+4v)$; 23. $(4u+\frac{1}{2}v)(\frac{1}{2}u-4v)$; 24. $(4u+\frac{1}{2}v)(\frac{1}{2}u+4v)$.

I.6 Research Problems

Some students find it very stimulating to be assigned small research projects in which, for example, rules and principles can be found and supported.

First Example

A deeper look into the rules of number sequence can begin with the question: How can both factors of an end product change without changing the value of the product?

Let us start with

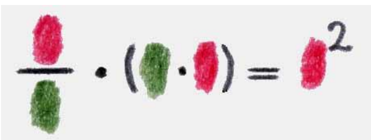
$$10 \cdot 10 = 100$$

and halve the first factor 10 so that we get 5. In order for the product to remain the same, the second factor must be doubled to 20:

$$\frac{10}{2} \cdot (2 \cdot 10) = 5 \cdot 20 = 100$$

This applies to any products that have two equal factors: If the first factor is halved, then the second factor must be doubled so that we get the same result. If the first factor is cut in thirds, then the second factor must be tripled, and so forth.

Generally, the rule of equal products as it has been used in the above example, can be represented using colors:

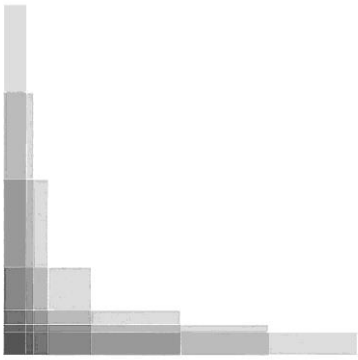


With the help of letters it has this form:

$$\frac{a}{c} \cdot (c \cdot a) = a^2$$

Converting factors with the same product can be looked at in many different ways. For instance, two students can stand about 1 meter apart, and a third student is placed in the middle. The third student halves the distance between the first two. A fourth student stands in a place where this distance is doubled. Now, one of the first two students slowly goes up and down the line on which the other two are standing. Students three and four must pay attention that they keep the correct position to represent the halving or doubling! While Student three is moving slower, student four must move twice as fast. The rule can be experienced in the relationships of the speeds. The difference would be just as pronounced if using thirds, or rather, tripling.

Another possibility would be to represent the product – starting from the square – as a rectangle.



Which rules apply to any end product a · b, and how can they be represented?

I.6.1 Second Example

Calculate the following products and compare the results.

10 ·10

9 ·11

8 ·12

7 ·13

6 ·14

5 ·15

4 ·16

3 ·17

2 ·18

1 ·19

0 ·20

Here, the *products* change while the *sums* of the factors always remain the same. The results are a sequence of numbers whose differences (first *differences sequence!*) happen to be the sequence of odd numbers!

$10 \cdot 10 = 100$

1

$9 \cdot 11 = 99$

3

$8 \cdot 12 = 96$

5

$7 \cdot 13 = 91$

7

$6 \cdot 14 = 84$

9

$5 \cdot 15 = 75$

11

$4 \cdot 16 = 64$

13

$3 \cdot 17 = 51$

15

$2 \cdot 18 = 36$

17

$1 \cdot 19 = 19$

19

$0 \cdot 20 = 0$

Here is a good place to insert a very nice observation for those students who have an especially agile mind.[33]

If we have this equation:

$$(10 \cdot 10 - 9 \cdot 11) + (9 \cdot 11 - 8 \cdot 12) + (8 \cdot 12 - 7 \cdot 13) + ... + (2 \cdot 18 - 1 \cdot 19) + (1 \cdot 19 - 0 \cdot 20),$$

By calculating the individual parentheses we get, on the one hand, the sum:

$$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19$$

On the other hand, the sum is reduced because all the other numbers are reciprocally raised to $10 \cdot 10 - 0 \cdot 20 = 100$. It is:

$$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 = 100$$

What is the result if we start with 12 instead of 10? We get:

$12 \cdot 12 = 144$

1

$11 \cdot 13 = 143$

3

$10 \cdot 14 = 140$

5

[33] I have Dieter Volkert to thank for the tip about this observation.

$$
\begin{array}{rl}
9 \cdot 15 = 135 & \\
& 7 \\
8 \cdot 16 = 128 & \\
& 9 \\
7 \cdot 17 = 119 & \\
& 11 \\
6 \cdot 18 = 108 & \\
& 13 \\
5 \cdot 19 = 95 & \\
& 15 \\
4 \cdot 20 = 80 & \\
& 17 \\
3 \cdot 21 = 63 & \\
& 19 \\
2 \cdot 22 = 44 & \\
& 21 \\
1 \cdot 23 = 23 & \\
& 23 \\
0 \cdot 24 = 0 &
\end{array}
$$

The same difference sequence occurs as in the previous case! We can try this using other beginning numbers, continue the reduction into the area of negative numbers, increase the differences by 2, vary the beginning numbers, etc. There is a wide field for experimentation.

In order to see that the differences sequences really are dependent upon the beginning number, we can enlist the help of an algebraic illustration: *a* can be any number for the beginning. We build the consecutive products:

$$
\begin{array}{rl}
a \cdot a = (a - 0)(a + 0) = a^2 - 0 & \\
& 1 \\
(a - 1)(a + 1) = a^2 - 1 & \\
& 3 \\
(a - 2)(a + 2) = a^2 - 4 & \\
& 5 \\
(a - 3)(a + 3) = a^2 - 9 & \\
& 7 \\
(a - 4)(a + 4) = a^2 - 16 & \\
& 9 \\
(a - 5)(a + 5) = a^2 - 25 & \\
& 11 \\
(a - 6)(a + 6) = a^2 - 36 & \\
\ldots\ldots\ldots & \ldots\ldots
\end{array}
$$

We can see that the third binomial formula plays the essential role. By changing the factors each time by 1, we get the sequence of square numbers, and their differences give the sequence of odd numbers. The other cases spoken of can be investigated in a similar way. This algebraic illustration proves to be an effective way to gain insight into rules of coherency that would otherwise be difficult to understand.

Third Example

We have $5 \cdot 5 = 25$, $15 \cdot 15 = 225$, $25 \cdot 25 = 625$, $35 \cdot 35 = 1225$,… While building the result, try and find a rule.

Solution:

The 5-numbers that are multiplied by themselves have the form (10a + 5), whereby the numbers 0, 1, 2, 3… should be put in for *a* in that order. We multiply and get:

$$(10a + 5)\cdot(10a + 5) = 100a\cdot a + 50a + 50a + 25 = 100a\cdot a + 100a + 25$$

$$= 100(a + 1)a + 25$$

This last expression shows the formation of the result: It is always a multiple of the number 100. 25 is to be added to it. 25 must always be at the end. Also, this rule allows us to calculate many such products in our head. For instance, if we have to multiply $85 \cdot 85$, then we calculate $9 \cdot 8 = 72$, $100 \cdot 72 = 7200$, $7200 + 25 = 7225$, and there we have the result.

We can use the result of this problem to calculate products of the following kind:

$83 \cdot 87$, $41 \cdot 49$, $74 \cdot 76$, generally:

$$[(10a + 5) - b][(10a + 5) + b] = 100(a + 1)a + 25 - b^2.$$

Example: $83 \cdot 87$: a = 8, b = 2. We calculate $100 \cdot 9 \cdot 8 = 7200$, $7200 + 25 = 7225$, $7225 - 4 = 7221$

Fourth Example

Which numbers can occur as the last digit of a square number?

Fifth Example

Which number pairs can occur as the last two digits of a square number?

Sixth Example

When two numbers are either symmetrical with 25 or are multiples of 25, then the squares of those numbers concur in their last two digits.

Example: $17^2 = 289$, $33^2 = 1089$ oder $64^2 = 4096$, $86^2 = 7396$. [1]

6. Recursive Arithmetic

The following exercises will provide practice in a thinking form that is interesting by itself, but has also gained in significance through computer technology. We are talking about so-called *recursive arithmetic*. Sequences of numbers are calculated by a method in which the calculation presupposes the preceding numbers gotten by the same method. In the upper grades such *recursive calculations* of numbers can be studied more systematically. Here, they offer good practice for dealing with negative numbers and recognizing patterns in sequences of numbers.

Let us begin with a few easy examples. We will divide the numbers in a sequence with a

[1] Die letzten beiden Beispiele verdanke ich einem Hinweis von Gerhard Kowol

semi-colon to avoid confusion.

1. Example

The construction principle of a number sequence shall be: Add 2 to a number in a number sequence in order to get the next number.

Naturally, some *beginning number* must be given so that the next one can be calculated. For instance, if we choose 1 as the beginning number then the next number is 3. The following numbers are 5; 7; 9… If we begin with 4 then the following number is 6, then 8; 10; 12…

2. Example

Multiply the preceding number by 2 and add 1.

If we choose 0 as the beginning number then we get this number sequence:

$0; 2 \cdot 0 + 1 = 1; 2 \cdot 1 + 1 = 3; 2 \cdot 3 + 1 = 7; \ldots$, that is, 0; 1; 3; 7; 15; 31;...

If we begin with -1 we get:

$2 \cdot (-1) + 1 = -2 + 1 = -1$

The next number is calculated exactly the same. So, all the numbers in the sequence remain -1.

If we begin with -2 we get:

$2 \cdot (-2) + 1 = -4 + 1 = -3; 2 \cdot (-3) + 1 = -6 + 1 = -5; 2 (-5) + 1 = -9; 2 (-9) + 1 = -17;$ ... *that is* –2; -3; -5; -9; -17….

Using the same construction principle one can get very different numbers sequences depending upon what number one begins with.

3. Example

Calculate the numbers in the sequence by multiplying the preceding number by -1 and then adding 5.

We begin with the number 0 and we get:

$-1 \cdot 0 + 5 = 5; -1 \cdot 5 + 5 = 0;$

The number sequence we get is:

5; 0; 5; 0…

If we begin with 1 instead of 0 we get:

$1; \; -1 \cdot 1 + 5 = -1 + 5 = 4; \; -1 \cdot 4 + 5 = -4 + 5 = 1; \; -1 \cdot 1 + 5 = -1 + 5 = 4; \; -1 \cdot 4 + 5 = 1;$ *that is*

1; 4; 1; 4; 1 ...

If we begin with -1 we get:

$-1 (-1) + 5 = 1 + 5 = 6; \; -1 \cdot 6 + 5 = -6 + 5 = -1; \; -1 (-1) + 5 = 1 + 5 = 6$, *that is*

-1; 6; -1; 6; -1; ...

If we begin with -2 we get:

$-2; \; -1 (-2) + 5 = 2 + 5 = 7; \; -1 \cdot 7 + 5 = -7 + 5 = -2; \; -1 (-2) + 5 = 2 + 5 = 7;$

that is *-2; 7; -2; 7; -2; ...*

4. Example

Calculate the numbers in the sequence by multiplying the preceding number by -2 and then subtracting 5.

If we choose the beginning number as 1, then we get the following sequence:

$-2 \cdot 1 - 5 = -7; -2\,(-7) - 5 = 14 - 5 = 9;\; -23; 41; -87; \ldots$

If the beginning number is -1 we get:

$-3; 1; -7; 9; -23; \ldots$

If the beginning number is 0 we get:

$-5; 5; -15; 25; -55; \ldots$

Calculating fractions can also be practiced with such recursively determined number sequences and it should be done often.

5. Example

In order to calculate a number in a sequence, divide the preceding number in half. If we begin with the number 8 we get the following sequence:

8; 4; 2; 1; ½; ¼…

With the beginning number of 5 we get: $5/2;\ 5/4;\ 5/8;\ 5/16;\ldots$

6. Example

Calculate a number in a sequence by dividing the preceding number in half and then adding 1/3.

With the beginning number of 1 we get:

$1;\ 1/2 \cdot 1 + 1/3 = 5/6;\ 1/2 \cdot 5/6 + 1/3 = 3/4;\ 1/2 \cdot 3/4 + 1/3 = 17/24;\quad 1/2 \cdot 17/24 + 1/3 = 11/16$

Other beginning numbers such as 0 or $-1/2$ can also be chosen.

According to one's own competence and the ability of the class, at this point one could try to express such principles of numbers sequences algebraically. This could occur in the following way, for example: We think of the numbers in a sequence as being designated by a letter. However, if one wanted to use a, b, c, d…, for example, first of all, one could not go further than to the end of the alphabet, and second, it would not be immediately apparent how far down the line the *q* is, for example. Therefore, another way was chosen: One designates all the numbers in a sequence with the same letter, such as *a* or *b* or …, and puts a number written in subscript next to the letter that lets us know where in the sequence that number is.

For example $a_1; a_2; a_3; a_4; \ldots$

One can describe any number with a_n. This is called the n^th^ number.

These subscript numbers are called indexes. Dealing with indexes requires a lot of concentration. They describe what position a number has in a sequence. When one sees a_7 it means that *a* is the seventh number in the sequence.

As has already been mentioned, one prefers to write a generic number in a sequence as a_n. Then the preceding number, if there is one, would be a_{n-1}. Pay special attention here: a_{n-1} stands for the number in the series that precedes the number a_n and *not* the number $a_n - 1$. If one were to calculate $a_n - 1$ then one would subtract 1 from the *nth* number and would normally *not* get the preceding number. In general, the same is true in that a_{n+1} is different than $a_n + 1$.

With indexes, the additions and subtractions indicate their positions relative to one another and not the value of the numbers themselves. The following line describes the numbers to each side of $a_{n:}$

...; a_{n-2}; a_{n-1}; $\boldsymbol{a_n}$; a_{n+1}; a_{n+2}; ...

One can very nicely express the principles of numbers sequences using this written form. We will go through the above examples once again. Once the stated principle has been found, the students should calculate multiple numbers in the sequence with a beginning number that is given.

1. *Example:* The construction principle of a number sequence shall be: Add the number 2 to the numbers in a sequence to get the next term.

Solution: Written in algebraic form, this construction principle is:

$a_{n+1} = a_n + 2$

2. *Example:* Multiply the previous number by 2 and add 1.

Solution: $a_{n+1} = 2a_n + 1$

3. *Example:* Calculate a number in a sequence by multiplying the previous number by -1 and adding 5.

Solution: $a_{n+1} = -a_n + 5$

4. *Example:* Calculate the number in a sequence by multiplying the previous number by -2 and adding 5.

Solution: $a_{n+1} = -2a_n – 5$

5. *Example:* In order to calculate a term in a number sequence, halve the previous term.

Solutio: $n\, a_{n+1} = \frac{a_n}{2}$

6. *Example:* In order to calculate a term in a number sequence, halve the previous number and add 1/3.

Solution $a_{n+1} = \frac{a_n}{2} + \frac{1}{3}$

One can also connect two number sequences so that the one can only go further if the other also makes a step forward. Something like this can also be compared to certain human relationships. An example of such a limited number sequence is shown by the following rule:

$a_n = a_{n-1} + b_{n-1}$

$b_n = a_{n-1} – b_{n-1}$

If we begin with the beginning values of $a_1 = 1$ and $b_1 = 1$, we get this number sequence:

1; 2; 2; 4; 4; …

1; 0; 2; 0; 4; …

If we begin with the beginning values of $a_1 = b_1 = ½$, we get this number sequence:

½; 1; 1; 2; 2; …

½; 0; 1; 0; 2; …

With beginning values of $a_1 = 1$, $b_1 = -1$, we get:

1; 0; 2; 0; 4; 0; 8; 0; …

-1; 2; -2; 4; -4; 8; -8; 16; …

Again, there is a surprising variety in the formation of number sequences dependent upon the beginning values. Here, we are dealing with mathematical construction principles of the simplest kind, but perhaps we can see something like a precursor of living formative principles: How differently can plants develop from the same seed, depending upon conditions under which their formative principles took effect. But, in spite of their variations, one still can recognize the same active principle. It has been told about a French princess that she instructed a gardener to find two leaves that were just alike in a nearby park. He was not successful. And yet, this sameness is what is normally of interest to us with trees; even more so than their varied characteristics – which does not mean that one can not be interested in each individual form. Many children love very special trees with their familiar forms. But this is always in effect: It is a birch, or an oak, or some other tree. This relationship between individual form and general form principle can lead us to this question: Can we also find a formative principle in a given number sequence?[35]

We must limit ourselves to a few very simple examples. Let us take the number sequence 3; 5; 7; 9…, for example.

It is easy to see that the next number in the sequence is arrived at by adding the number 2 to the preceding number. Using letters, the principle is expressed like this:

$$a_n = a_{n-1} + 2$$

In our example the beginning value is 3. If we choose another beginning value, such as $-\frac{1}{2}$, for instance, then according to the same principle we get this sequence: $-\frac{1}{2}$; $\frac{3}{2}$; $\frac{7}{2}$; $\frac{11}{2}$… We get completely different numbers in spite of using the same construction principle. But it is also easy to find the relationship between the two sequences. We only need to take away two consecutive numbers and in both cases we always get the same difference of 2: $a_{n+1} - a_n = 2$ or $a_{n+1} = a_n + 2$

Another example:

1; 2; 4; 8; 16…

Here we see that the next number is arrived at by doubling the preceding number, that is, by multiplying it by 2. This is the principle:

$$a_n = 2a_{n-1}$$

The beginning value is $a_1 = 1$. If we choose 1.5 as the beginning value we get this sequence:

1.5; 3; 6; 12; 24…

Since the same principle is in effect here, we recognize that if we divide two consecutive numbers by each other, the following applies to both sequences:

$$a_n : a_{n-1} = 2$$

Besides the general benefit of understanding such principles, these exercises provide an opportunity to review working with fractions, decimal fractions, negative numbers, or also, if

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[35] From a finite amount of given numbers one cannot clearly determine a principle if nothing further is known about the number series. There can be many different ones that produce digressive numbers after the given numbers. But here, we always assume the "simplest" case.

one uses large numbers, written mathematical operations.

The following exercises are intended as suggestions. One can easily think of other examples, or one can ask students to think of their own examples.

Practice 41

1. In the following, the construction principle for various number sequences is given. Express it in words and calculate the first seven numbers with the given three beginning values for a_0:

a) $a_n = 4a_{n-1} + 2$; $a_0 = 1/-1/0$. b) $a_n = (a_{n-1} - 1)^2$; $a_0 = 0/1/2$. c) $a_n = a_{n-1} + n$; $a_0 = 0/1/2$. d) $a_n = - n \cdot a_{n-1}$; $a_0 = 0/1/-2$.

2. In the following, the first numbers in a sequence are given. Determine the construction principle in words and in a formula, if possible:[36]

b) *1; 3; 5; 7; ...*b) *1; 2; 4; 8; 16; …*c) *3; 6; 12; 24; 48; ...*d) 1; 2; *4; 7; 11; 16; 22; ...*e) *1; 3; 6; 11; 20; 37; 70; …*(Difficult!)

3. Parking fees: The first hour costs $2.50; every following hour costs $1.50. How much does one pay after 2, 3, 4… hours? Can you write a formula using C for the cost and h for the number of hours?

4. A child saves $5.00 every month from their allowance. The parents promised that at the end of the year they would match 1/10th of the child's savings. How will the child's savings grow if they save for several years without taking anything out?

5. How much real buying power does the child in problem 4 have at the end of the 1st, 2nd, 3rd … year if they live in a country where the annual inflation rate is 10%?

6. At a jumping procession in a certain village, the people always go 3 steps forward and 2 steps back. How many steps until they have gone 10 meters if one step is about 50 centimeters long?

Solutions:

1.

a) (1) 1; 6; 26; 106; 426; 1706; 6826; … Why is there always a 6 at the end after a_1 ? (2) –1; -2; -6; -22; -86; -342; -1366; … (3) 0; 2; 10; 42; 170; 682; 2730; … Pay attention to the end numbers!

b) (1) 0; 1; 0; 1; 0; 1; 0; … (2) 1; 0; 1; 0; 1; 0; 1; … (3) 2; 1; 0; 1; 0; 1; 0; …) Also try the sequence for $a_0 = 3$!

c) (1) 0; 1; 3; 6; 10; 15; 21; … (2) 1; 2; 4; 7; 11; 16; 22; … (3) 2; 3; 5; 8; 12; 17; 23; …

d) (1) 0; 0; 0; 0; 0; … (2) 1; -1; 2; -6; 24; -120; 720; … (3) –2; 2; -4; 12; -48; 240; -1440; …

2.

a) To find the next number in the sequence add 2 to the preceding number. Begin with 1. $a_n = a_{n-1} + 2$; $a_0 = 1$.

b) To find the next number in the sequence multiply the preceding number by 2. Begin with 1. $a_n = 2 \cdot a_{n-1}$; $a_0 = 1$.

c) To find the next number in the sequence multiply the preceding number by 2. Begin with 3. $a_n = 2 \cdot a_{n-1}$; $a_0 = 3$.

d) To find the next number in the sequence add to the preceding number the index number of the new one. Begin with 1. $a_n = a_{n-1} + n$; $a_0 = 1$.

[36] See the previous annotation

e) To find the next number in the sequence multiply the preceding number by 2 and subtract the index number which is now 2 less. Begin with 1. $a_n = 2 \cdot a_{n-1} - (n-2)$; $a_0 = 1$

3. 2 hours \$4.00; 3 hours \$5.50; 4 hours \$7.00; h hours K = \$2.50 + (h-1) · \$1.50

4. After one year of saving a child has accumulated \$66.00; after two years \$132.00; after 3 years \$198.00; after 4 years \$264.00; after n years n · \$66.00.

5. As much as was paid in by the child because inflation cancelled out the payments made by the parents.

6. They reach the 10-meter mark for the first time after a total of 88 steps, and after 96 steps they no longer fall behind the 10-meter mark.

I.7 Research Problems

Practice 42

1. Do the previous problem 4 with the parents always contributing 1/10th of the amount saved at the end of each year. Calculate the first 4 years.

2. The principle for a number sequence is: $3a_n = a_{n-1} + 12$. Check to see how many numbers of the first 10 are prime numbers if $a_0 = 1, 3, 5, 7, 11$. Remember not to count 1 as a prime number.

3. The construction principle for a number sequence is: $a_n = a_{n-1} + a_{n-2}$. One must know two previous numbers in order to get the next number. Therefore two beginning numbers must be given. If one puts $a_0 = 1$ and $a_1 = 1$ then one gets the famous *Fibonacci sequence*; 1,1,2,3,5,8,13,21… These numbers play an important role in nature in positioning of leaves and many other things.

Assignment: Calculate the first 12 numbers using the given construction principle, but different beginning numbers. Choose, for example, $a_0 = 1$; $a_1 = 7$; or $a_0 = -3$; $a_1 = 101$. With the help of a calculator, determine the number sequence: $b_n = a_n : a_{n-1}$. Do you observe something conspicuous?

4. Calculate 10 numbers in the sequence $a_n = k \cdot a_{n-1} \cdot (1 - a_{n-1})$ using any beginning value a_0 between 0 and 1, like $a_0 = 0.8$, for example. Choose k = 2, k = 3.2, and k = 4. What can be observed?

Solutions of Research Problems:

1. After one year \$60 + 1/10 · \$60,- = \$66,-; after two years \$66,- + \$60,- + 1/10 · (\$66,- + \$60,-) = \$126,- + \$12,60 = \$138,60 **;** after three years \$138,60 + \$60,- + 1/10 · \$198,60 = \$218,46; after four years \$218,46 + \$60,- + 1/10 · \$278,46 = \$306,306 ≈ \$306,31

2. There are six prime numbers in $a_0 = 1$; one in $a_0 = 3$; eight in $a_0 = 7$; seven in $a_0 = 7$; seven in $a_0 = 11$.

3. In the first example one gets the number sequence 1; 7; 8; 15; 23; 38; 61; 99; 160; 259; 419; 678; …and as quotients b_n: (rounded to three numbers) 7; 1,143; 1,875; 1,533; 1,652; 1,605; 1,623; 1,616; 1,619; 1,618; 1,618; 1,618; …

In the second example one gets –3; 101; 98; 199; 297; 496; 793; 1289; 2082; 3371; 5453; 8824; … and as quotients b_n: –33,667; 0,970; 2,031; 1,492; 1,670; 1,599; 1,625; 1,615; 1,619; 1,618; 1,618; … It is interesting to note that, *regardless of the beginning numbers,* (except $a_0 = a_1 = 0$) the quotient sequence gets closer and closer to the golden ratio, G = 1,618.

4. For $a_0 = 0.8$ and k = 2 we get (rounded to three decimals): 0.320; 0,435; 0,492; 0.500; 0,500… The numbers in the sequence get closer to the value 0.500.

For $a_0 = 0.8$ and k = 3.2: 0,512; 0,800; 0,512; 0,800; 0,512; 0,800; … The numbers in the sequence get closer to the *two* values 0.513… and 0.799…

For $a_0 = 0.8$ and $k = 4$: 0,640; ≈0,922; ≈0,289; ≈0,822; ≈0,585; ≈0,971; … The numbers in the sequence do not get recognizably close to a value.