(Here are problems with the format)

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New Numbers and their Meaning. How to Deal with Dept. Negative Numbers

I.1 Introduction

In this chapter you will learn a new quality of numbers. We call this quality *negative* and you will not only learn how to use them but also why they are introduced and what a specific character they have.

Of course you already know some different kinds of numbers like *natural* numbers (1, 2, 3,…) the special number *zero*, *fractions*, *decimals*, and maybe you did already learn something about *negative* or even *rational* and *irrational* numbers. There are still more kinds of numbers that are studied in mathematics. Of course, we may ask many questions: How can one introduce new numbers? Is it allowed just to invent a new type of numbers? Can we invent a new type of numbers? Would that be meaningful? Usually we experience mathematics in its wonderful lawfulness. How can something new be invented? Or is it discovered? Many of these questions are discussed among mathematicians or philosophers. The good message is: There is not a last answer that has to be learned. Have your own experiences and try to understand them. Let us use the negative numbers as an example of a more general problem: How can a concept become *meaningful*?

Therefore the concept of *negative numbers* should be introduced with the greatest of care. When it is done with real consciousness, and not in a superficial way using simple geometric pictures, such as a number line, for instance, has far-reaching consequences. It is the concept of *less than nothing*[22]

But, first of all, let us make clear, that *negative* has the opposite *positive*. Through the words *positive* and *negative*, we are expressing qualitative polarities that, in normal life, are usually meant in a moral sense. We may speak of having a positive or negative attitude or also of positive or negative development. Such colorations of meanings do *not* have anything to do with the realm of numbers. Also, the positive or negative grades on a temperature express a polarity in relationship to a conventionally agreed-upon zero point, but it does not really give us a concept of negative numbers in the mathematical sense. Where do we have reason to expand the realm of known numbers, that is, the natural numbers, zero, and fractions, etc?[23]

Let us be clear that when one counts things only the natural numbers really make sense. If we enter a room and direct our attention to the chairs it contains then it would be one, two, three chairs. Even the zero is not used when counting because if we were to make something like an inventory list of the chairs in a classroom we would never be finished counting things in a room that had nothing in it! An inventory of your (e.g. class) room including things with zero items would supposingly contain 0 elephants, 0 icebergs, 0 moons, etc. That does not make sense!

Fractions are used for quantities that we can count and measure; that means, their quantities are determined by their relationship to a whole.

How do we get to the point of forming a *negative number quality*? We can make a beginning with the chairs we spoke of before: If we have twenty-five chairs in a classroom and are expecting thirty parents for a conference that evening, then we are *missing* five chairs.

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22 See also *Zahlen, negative* in *Rudolf Steiner zur Mathematik*, compiled by Ursula Kilthau and Georg Schrader, published by the Paedagogischen Forschungsstelle, of the Bund der Freien Waldorfschulen, Stuttgart 1994. Are there English versions?

23 The following has to do with building a content-oriented concept of negative numbers and not a model for the algebraic structure of whole numbers.

These five chairs are still only *an inner picture*. They are not yet concretely there. We do not know what specific chairs we will get. But, in terms of an amount, they can already be determined to be a *deficit*. When five chairs are brought into the room, then, in a certain sense, they erase the concept of the five chairs. The need has been fulfilled.

This is a similar relationship as that between the intellectual work of an architect and the real work of construction, as we have described in booklet I: Intellectual work is indispensable for the orientation of the practical work. However, one normally finds no concrete suggestions for specific bricks, etc. in an architect's plan. Only “bricks” in general, with certain characteristics, are suggested, but usually nothing specific about an individual brick. So then, the construction erases the need for the work of the plans in a certain sense. This is a similar effect as bringing in the chairs to the classroom; the plans are also no longer needed.

As Rudolf Steiner indicated in his *National Economics Course,* mental and physical work have a relationship of polarity whereby equal things are not standing in opposition to each other, but rather things that are qualitatively and characteristically different.[24] There is a sequence of time present in which the thought comes *first, followed by* the materially concrete. The interplay of these two things is what creates the reality. Thoughtlessly piling up bricks would never lead to anything sensible. But, without the bricks, the most beautiful plans would remain just that: plans. So, thinking and willing must work together. Naturally, both these polarities can be narrowly intertwined and exchanged when it comes to performing practical work. This special, interchangeable relationship becomes clearer when we think of the work divided up among many people, as is usually the case in our modern world of division of labor.

When we had lessons about electricity we got to know about polarity. The equalization of polarities in the form of positive and negative charges is what allows for the varied effects of electricity. However, what is not so easy to observe in that example, as it is in the relationship between mental and physical work, is the different quality of both poles. It is there, but at low level, without a model, it is not easily explained or easy to prove through experience.

Another important example of a *qualitative* polarity is found in the two sides of a balance sheet. Within a certain perimeter, with fixed and short-term assets, we are dealing with concrete, material amounts which we use to conduct business, except for the accounts receivable. The passive (liability) side characterizes, as we have described in Booklet I, the obligations of the business to other persons. They are usually financial payables, that is, something that exists through contractual agreement, and not in objective physicality. Even though this characteristic does not apply in all cases – consider that the previously mentioned accounts receivable are part of the short-term assets – it does reflect an aspect of the relationships.

Even the relationship of money in general to real goods and services can be looked at in this sense.

I.2 Entry into Negative Numbers without a lot of Bookkeeping Knowledge

Our introduction to negative numbers will begin with the balance sheet in its simplest form (The simplest structure of a balance sheet can be found starting Booklet I on page xxx.). One can also begin here if the first chapter of Booklet I was not dealt with. We will begin with a very basic personal balance sheet for a child's money. If Booklet I has already been gone through, the following will seem very easy. In a more general sense the presented personal balance sheet is probably not the best example because, obviously, the obligations (debt) are merely consumer credit and not the obligations resulting from invested capital as we went into

24 Rudolf Steiner, *World Economy*. Complete Works, in the place cited (annotation 7).

in Booklet I.

Let us imagine a child named Iona whose personal finances are in the following state: In her pocket she has five dollars. She is owed seven dollars from her older brother. Unfortunately, these active entries also have obligations against them: The five dollars in her pocket were borrowed from her younger sister, and her grandmother loaned her five dollars quite some time ago. The balance sheet looks like this:

Balance Sheet for Iona November 17, 20..

Assets			**Liabilities**
Cash	$5.00	Loan from grandmother	$5.00
Account receivable from brother	$7.00	Loan from sister	$5.00
————————	———— –	net worth	$2.00
sum	$12.00	sum	$12.00

The discretionary net worth – the difference between the assets and liabilities- is only $2.00. Iona's financial situation does not look as rosy as one is led to believe by looking at the $5.00 in her pocket. It would be relatively easy to spend the five dollars because they are immediately available. They describe the available *liquidity*, the possibility of spending or having money at one's disposal, but they have *debt* obligations against them in the amount of $3.00, leaving only $2.00.

If Iona really does spend the five dollars now then her situation becomes significantly worse. Her net worth is no longer $2.00 but she is now $3.00 in *debt*. If we wish to enter this amount on the right side of the balance sheet, then we can not simply enter $3.00 as we have done before because these dollars have a new quality. They can not fill one's pockets but they have an effect nevertheless. For example, one effect they have is if Iona should have $3.00 again at some point in the future, she should not freely spend it but rather use it to pay off her debt. The $3.00 is immediately used up for the debt.

Balance Sheet for Iona November 17, 20..

Assets			**Liabilities**
~~Cash~~	~~$5.00~~	Loan from Grandmother	$5.00
Account Receivable Brother	$7.00	Loan from Sister	$5.00
————————	————	Net worth as Debt	$3.00
Sum	$7.00	Sum	$7.00

We have to differentiate: 3 regular dollars that can be spent, and 3 dollars of debt that can eat up the 3 regular dollars. In order to differentiate between these two qualities we will write the regular dollar amount in red, and the debt dollar amount in blue. The red numbers are called "positive" and the blue numbers are called "negative". (We must not be bothered by the fact that one usually speaks of being *in the red* when one is talking about debt. In that case, the red color is there to stand out. For our purposes, the polarity of positive and negative is better expressed through the use of red and blue.)

We will now calculate using these red and blue, positive and negative, numbers. The addition sign and the zero can be written in white on the blackboard; here it is black. We begin with a series of simple addition problems. Please notice that e.g. ***4*** is as *positive 4* and ***4*** is read as *negative 4*.

Interpret the following problems always by their meaning:

4 + ***4*** = ***8*** ***4*** + ***2*** = ***2***

4 + ***3*** = ***7***
4 + ***2*** = ***6***
4 + ***1*** = ***5***
4 + ***0*** = ***4***
4 + ***1*** = ***3***

4 + ***3*** = ***1***
4 + ***4*** = ***0***
4 + ***5*** = ***1***
4 + ***6*** = ***2***
4 + ***7*** = ***3***

For example, ***4*** + ***2*** means that someone has $4 and gets 2 more. ***4*** + ***2*** means that someone has $4 but then incurs $2 in debt by buying something for $2, for instance, but does not pay right away. The person now has 2 dollars at their disposal. ***4*** + ***6*** means that someone has $4 in her possession, but has incurred debts in the amount of $6 so that they are now in debt; have more debt than credit.

I had such encounters as these with available cash and debt as a parent with our neighborhood store. Often in the evening something had to be gotten quickly from the store but there was no available cash in the house. A child would be sent (without money) to the store to get the bread, or milk, or whatever was needed. The cashier would write our family name on a piece of paper and lay it by the cash register. Both parents and children were known as customers. So, by reason of our trustworthiness, we got a credit (credere = trust, faith). It happened many times that I would go to the store with $20.00 in my pocket, for example, buy $10.00 worth of fruit and be confronted with the debt we already had when I went to the cashier to pay. If I was counting on the $10.00 change, suddenly it was used to pay off the existing debt. This scenario was usually suggested by the cashier's friendly question: “Are you also paying for your family today?” There was usually hardly anything left of my $10.00 in change. The positive dollars encountered the negative dollars. The credit had been paid off but at the same time I lost buying power.

If debt is *added to* debt, blue numbers to blue numbers, then they also increase just as positive numbers when they are added together. For example: ***4 + 6 = 10***

When subtracting negative numbers one must be clear about what it means to take away debt. We begin first with another series of problems in which the number subtracted is reduced until we come into negative numbers:

4 – ***3*** = ***1***
4 – ***2*** = ***2***
4 – ***1*** = ***3***
4 – *0* = ***4***
4 – ***1*** = *?*

What does it mean when a blue number is subtracted? We clarify this on the balance sheet: *When a debt entry is taken away, the net worth is immediately increased by the same amount.* In order to see this, let us take a look at Iona's balance sheet once again, in which she owes her grandmother five dollars.

Balance Sheet of Iona on …

Assets			Liabilities
Cash	$5.00	Loan from Grandmother	$5.00
Account Receivable from Brother	$7.00	Loan from Sister	$5.00
————————	————	Net worth	$2.00
Sum	$12.00	Sum	$12.00

If Iona's grandmother says: “I will forgive your debt”, then the balance sheet is changed:

Balance Sheet of Iona on …

Assets			**Liabilities**
Cash	$5.00	~~Loan from Grandmother~~	~~$5.00~~
Account Receivable from Brother	$7.00	Loan from Sister	$5.00
————————	———	Net worth	$7.00
Sum	$12.00	Sum	$12.00

The net worth was increased by exactly $5.00. This would also happen if someone gave Iona five dollars:

Balance Sheet of Iona on…

Assets			**Liabilities**
Cash	$5.00	Loan from Grandmother	$5.00
Account Receivable from Brother	$7.00	Loan from Sister	$5.00
Gift	$5.00	Net worth	$7.00
Sum	$17.00	Sum	$17.00

The net worth, the amount that Iona really has at her disposal, was increased by the same amount as if her debt was forgiven. The difference is that the *balance sheet total* has increased. But, taking away debt has the same effect on net worth as a positive entry added to the active side of the balance sheet. *For this reason we calculate the subtraction of a blue (negative) number the same as the addition of a red (positive) number.*

Sometimes students will come back with this argument: "If I have money in my pocket and someone forgives my debt, I still don't have more money in my pocket than before and I can't buy any more or less." That is correct if one is looking at the money at one's immediate disposal and not at the entire financial situation. One must think in terms of balance! Someone who is thinking in terms of reality, five dollars in one's pocket means something different if one has five dollars debt than if there was no debt.

We calculate like this:

4 – ***1*** = ***5***

4 – ***2*** = ***6***

4 – ***3*** = ***7***

……..

To differentiate between the two opposing number qualities, the use of color is very effective.[25] However, in real life it is not practical because the colors must be constantly changed. Therefore, another method was devised.

Let us think again about how negative numbers come about. They come about when more is given out than is on hand, that is, when we have *less than nothing*. Mathematically expressed, they come about when we subtract from one number another number that is larger, so that the subtraction process can not be completed and an *excess amount of subtraction energy is left over.* This is comparable to extraordinary hunger that one has after a long mountain bike ride. One may have hunger enough for five toasts, but finds only two in the kitchen. One eats the two toasts but is still hungry for three more. These are three negative toasts, or one could say: Subtraction energy for three more toasts. The following series of

25 In Chinese mathematics colors really were used to differentiate: red = positive, black = negative (For example, Lin Hui in 300 A.D). However, there were not yet any really independent number qualities.

subtraction problems shows, again, the transition from positive to negative numbers:

4 – ***3*** = ***1***

4 – ***4*** = *0*

4 – ***5*** = ***1***

4 – ***6*** = ***2***

4 – ***7*** = ***3***

. . . .

Because the creation of negative numbers is connected with subtraction, one uses a minus sign (usually short) to designate the blue numbers. Instead of ***3***, one writes +3, or simply 3, and instead of ***3***, one writes -*3*. This sign designates *number quality* and should not be confused with the subtraction sign. That is why one finds two different keys on every calculator: The minus key for subtraction and the prefix key with the signs +/-, or (-).

When one wishes to put -3 in a calculator, one can either calculate 0 – 3 or one can press the 3 key and the prefix key.

Because two operation signs behind each other do not make sense and because the subtraction sign and the negative sign can easily be mixed, one has to mark a negative number with a negative sign in front and to include both into parentheses, unless it is positioned at the beginning of a term. Instead of **4** - **3** = **7** one puts 4 - (-3) = 7, but **4** – **3** = **7** can be written without colors and without parentheses: -4 – 3 = -7.

Naturally, we have the following equalities:

$1 + (-1) = (-1) + 1 = 0$

$2 + (-2) = (-2) + 2 = 0$

$3 + (-3) = (-3) + 3 = 0$

$4 + (-4) = (-4) + 4 = 0$

$5 + (-5) = \ldots$

In algebra we express this using letters as a general rule:

If a is a positive number then $-a$ is the opposite negative number. Therefore:

$a + (-a) = (-a) + a = 0$

But what is $-a$ if a is already a negative number? We have found the rule: If we subtract a negative number then we get the same result as when we add the opposite positive number:

$b - (-a) = b + a$

Certainly, when calculating positive and negative numbers we wish to arrive at $a - a = 0$, regardless of whether a is a positive or negative number. If a is negative then we have to add the opposite positive number in order to reach 0. For example if we have $a = -7$, that means

$a - a = -7 - (-7)$.

Subtracting the -7 is done by adding the opposite positive number:

$-7 - (-7) = -7 + 7 = 0$.

$+7$ is the opposite number of -7. Therefore, we can say, in general, that:

$-(-a) = a$,

regardless of whether a is positive or negative. We describe $-a$ as the *opposite* number of a, no matter if we start with a positive or negative number. Zero is actually its own inverse number:

$0 = -0$

The entire subject of positive and negative numbers plus the zero is described as the *whole* *numbers*. But because the terms *positive* and *negative* can be easily used with fractions and decimals (what is just a different way to write fractions), then the number area covered by fractions (and decimals) is also widened into the negative. All of the whole numbers and fractions together are called the *rational numbers*. The pure value of any number z is described as its *absolute value*. One writes it as |z|. For example, if z = -1.5, then |-1.5| = 1.5. If z is positive or zero, then |z| = z. If z is negative then |z| = -z (which is positive!).

Practice 1

First, the red and blue numbers that have been introduced will be used in simple problems. Positive numbers are **red**, and negative numbers are **blue**.

1. ***1*** + ***1*** =
2. ***2*** – ***2*** =
3. ***3*** – ***3*** =
4. ***4*** + ***4*** =
5. ***5*** – ***5*** =
6. ***5*** – ***5*** =
7. ***1*** – ***2*** + ***3*** – ***4*** + ***5*** – ***6*** +***7*** – ***8*** + ***9*** =
8. ***1*** + ***2*** – ***3*** + ***4*** – ***5*** + ***6*** – ***7*** + ***8*** – ***9*** + ***10*** =
9. ***1*** + ***2*** – ***3*** – ***4*** +***5*** + ***6*** – ***7*** – ***8*** + ***9*** – ***10*** =
10. ***1*** – ***9*** + ***2*** – ***8*** + ***3*** – ***7*** + ***4*** – ***6*** + ***5*** + ***5*** – ***6*** + ***4*** – ***7*** + ***3*** – ***8*** + ***2*** – ***9*** + ***1*** =
11. ***13*** – ***26*** – ***39*** =
12. ***1*** – ***2*** + ***3*** – ***4*** + ***5*** – ***6*** + ***7*** =
13. ***2*** – ***4*** + ***6*** – ***8*** + ***1 0*** – ***12*** + ***14*** =
14. ***3*** – ***6*** + ***9*** – ***12*** + ***15*** – ***18*** + ***21*** =
15. ***4*** – ***8*** + ***12*** – ***16*** + ***20*** – ***24*** + ***28*** =

Anwers:
1) ***2***, 2) ***0***, 3) ***6***, 4) ***0***, 5) ***10***, 6) ***0***, 7) ***5***, 8) ***5***, 9) ***1***, 10) ***0***, 11) ***0***, 12) ***4***, 13) ***8***, 14) ***84***, 15) ***112***.

Practice 2

Here we will practice writing the notation with algebraic signs. If one of these problems should contain parentheses, it is recommended that one first remove them, that is, calculate them as follows, for example:

1– (–2) = 1 + 2 = 3.

(Careful, the problems and their solutions are only partly the same as those in exercise one!)

1. *1 + 1 =*
2. *2 – 2 =*
3. *3 – (–3) =*
4. *–4 + 4 =*
5. *–5 – 5 =*
6. *–5 – (–5) =*
7. *1 – 2 + 3 – 4 + 5 – 6 +7 – 8 + 9 =*
8. *–1 + 2 – 3 + 4 – 5 + 6 – 7 + 8 – 9 + 10 =*
9. *1 + (–2) – 3 – (–4) +5 + (–6) – 7 – (–8) + 9 – 10 =*
10. *1 – 9 + 2 – 8 + 3 – 7 + 4 – 6 + 5 – 5 + 6 – 4 + 7 – 3 + 8 – 2 + 9 – 1 =*
11. *13 – (–26) – 39 =*
12. *a + a =*
13. *b – b =*
14. *c + (-c) =*
15. *d – (-d) =*
16. *1 – 2 + 3 – 4 + 5 – 6 + 7 =*
17. *2 – 4 + 6 – 8 + 1 0 – 1 2 + 14 =*
18. *3 – 6 + 9 – 12 + 15 – 18 + 21 =*
19. *4 – 8 + 12 – 16 + 20 – 24 + 28 =*
20. *a – 2a + 3a – 4a + 5a – 6a + 7a =*

21. Make use of the answers of problem 20. for problems 16. to 19.

22. $a - (-a) - 2a =$

23. $2a - (-3a) + b - (-2b) =$

24. $-b + a - 2b + 2a + 3b - 3a =$

Anwers:
1) *2*; 2) *0*; 3) *6*; 4) *0*; 5) *–10*; 6) 0; 7)*5* ; 8) *5*; 9) *-1*; 10) *0*; 11) *0*; 12) *2a*; 13) *0*; 14) *0*; 15) *2d*; 16) *4*; 17) *8*; 18) *12*; 19) *16*; 20) *4a*; 21) The numbers 1, 2, 3, 4 should be consecutively inserted for *a*. This will give the proper result like in problems 16. – 19.; 22) *0*; 23) *5a + 3b*; 24) *0*.

Practice 3

The following table documents John's income and expenses over a longer period of time. In this way he can maintain control of his money. This is not really correct bookkeeping in the strictest sense but it does help him to keep an overview of his finances. Look at the effect of each transaction on the cash and net worth. Purchases are considered to be consumption and not investments. That is why their value is not listed on the balance sheet.
This is how John's balance sheet looked on January 12th:

Balance Sheet of Iona on …01.12.

Assets			**Liabilities**
Cash	$22.00	Loan from Sister	$2.70
Loan to Hans	$5.00	Net worth	$24.30
Sum	$27.00	Sum	$27.00

The following transactions take place. It is shown how each transaction affects the cash balance and the net worth. Look at each transaction!

Date	From/To	Income	Expense	Cash on Hand	Net Worth
12-02	Beginning Balance			22.00	24.30
12-03	Gift from Grandma	5.00		27.00	29.30
12-03	Purchase		2.30	24.70	27.00
12-03	Loan to Fred		7.00	17.70	27.00
12-05	Purchase		12.42	5.28	14.58
12-06	Loan from Klaus	15.00		20.28	14.58
12-12	Bicycle Repair		8.50	11.78	6.08
12-12	Gift for Luisa		9.80	1.98	-3.72
12-12	Loan Repayment from Hans	5.00		6.98	-3.72
12-13	Book Purchase		6.50	0.48	-10.22
15-12	Allowance (Gift)	15.00		15.48	4.78

Iona's balance sheet on October 1st:

Balance Sheet of Iona on …01.10.

Assets		Liabilities	
Cash	$30.00	Loan from Sister	$7.00
Loan to Francisca	$5.00	Net Worth	$28.00
Sum	$35.00	Sum	$35.00

Complete the entries for cash on hand and net worth in the following table!

Date	**From/To**	**Income**	**Expense**	**Cash**	**Net Worth**
10-2	Beginning Balance			30.00	28.00
10-3	Earned	10.00			
10-4	Purchase		1.70		
10-4	Repayment to Sister		7.00		
10-6	Purchase		12.42		
10-8	Loan to Francisca		15.00		
10-12	Loan from Grandma	25.00			
10-12	Gift for Luisa		19.80		
10-13	Repayment of both Loans from Francisca	20.00			
10-13	Book Purchase		22.00		
10-15	Allowance (Gift)	15.00			

What is Iona's net worth on 10-15?

Solutions:

Be aware that a loan given or taken does not change the balance of net worth but it does change the balance of cash on hand. It would appear that John and Iona both handle their money responsibly. They do not incur large debt and they keep an eye on their financial situation.

Iona's entries lead to the following results:

Date	**From/To**	**Income**	**Expense**	**Cash**	**Net Worth**
10-2	Beginning Balance			30.00	28.00
10-3	Earnings	10.00		40.00	38.00
10-4	Purchase		1.70	38.30	36.30
10-4	Repayment of Loan to Sister		7.00	31.30	36.30
10-6	Purchase		12.42	18.88	23.88
10-8	Loan to Francisca		15.00	3.88	23.88
10-12	Loan from Grandma	25.00		28.88	23.88
10-12	Gift for Luisa		19.80	9.08	4.08
10-13	Francisca Repays both Loans	20.00		29.08	4.08
10-13	Book Purchase		22.00	7.08	-17.92
10-15	Allowance (Gift)	15.00		22.08	-2.92

Iona has debt. The fact that she still possesses some liquidity is due to the loan from her grandmother.

I.3 The System of Positive and Negative Numbers

With the natural numbers we speak of larger and smaller numbers, but we mean actually the *more* or *less* that is indicated by the relationships of the numbers. 8 things are more than 5 things, or 9 is more than 8. We also say: *8 is greater than 5 and 9 is greater than 8.* One writes it as 8 > 5 and 9 > 8. (Read that: 8 *greater than* 5 and 9 *greater than* 8.) The relationship of the numbers expressed in the reverse would be: 5 < 8 and 8 < 9. (Read that: 5 *less than* 8 and 8 *less than* 9.)

If 9 > 8 and 8 > 5, then also 9 > 5. If 5 < 8 and 8 < 9, then also 5 < 9.

In general one can say:

If a > b and b > c, then a > c

also:

If a < b and b < c, then a < c

I.1.1 The order of numbers

Practice 4

1. Put the numbers in order from the smallest to the largest: 75, 13, 17, 1, 100

Solution: 1, 13, 17, 75, 100

All positive numbers, including fractions, can be ordered in the normal way.

2. Put the numbers in order from the smallest to the largest: 1; 0.6; 1.2; 4/7; 4/9; 5/9; 2.

Solution: 4/9; 4/7; 0.6; 5/9; 1; 1.2; 2

How do the negative numbers fit into the order of the positive numbers? First, we must remember that the number 0 (zero) is less than every positive number. It is *smaller* than every positive number. As we have discussed, a negative number describes *less than nothing*. That is why all negative numbers are *smaller* than zero.

For example, compare the numbers -5, -8, and -9. -8 is *less than* -5, and -9 is *less than* -8. Just as with the positive numbers, we write it like this: -9 < -8 and -8 < -5. Clearly stated: 9 is *larger* than 8, but -9 is *smaller* than -8. It is seen this way also when speaking of debt: If someone has $9.00 in debt, then they have *less* assets than someone who has $8.00 in debt.

In general, we can say:

If a > b, then -a < -b

This should be thoroughly discussed and practiced. Of course a person with $9.00 debt has more debt than a person with $8.00 debt. In this case, the basis for comparison is not the assets, but the debt.

Practice 5

Put the following numbers in order from the smallest to the largest:

1) $3; -5; 17; -19; \frac{1}{2}; -\frac{1}{3}; -2; 0; -\frac{3}{10}; 100; -100; -\frac{16}{3}; -\frac{21}{4}; -\frac{26}{5}; -\frac{17}{19}.$

2) *2; 0; 10; -2,22; -3,33; 2,17; -2,43; -122,21; -0,01; -1,11; 0; 0,02; -2000; 10.*

Solution:

1) $-100; -19; -\frac{16}{3}; -\frac{21}{4}; -\frac{26}{5}; -5; -2; -\frac{17}{19}; -\frac{1}{3}; -\frac{3}{10}; 0; \frac{1}{2}; 2; 3; 17; 100.$

2) *-2000; -122,21; -3,33; -2,43; -2,22; -1,11; -0,01; 0; 0,02; 0,10; 2,17; 10.*

See how much easier it is to order decimal fractions than regular fractions!

I.4 Using Negative Numbers

Our introduction to negative numbers began with one of their most important uses; calculating credit and debt. Once one has learned how to use negative numbers they can be used to great advantage for many different applications. Following are some examples that can be expanded upon:

I.1.2 Weight and buoyancy (Boost?)

If an object goes under water then the *weight* works toward the bottom and the *boost* works toward the top. If the weight (W) is greater than the boost (B), the object will continue to sink, but if the boost is greater than the weight, the object will come to the top. It rises. If both forces are equal it floats. One can describe both cases, sinking and rising, with two formulas. If W > B then $R_w = W - B$ is the *remaining weight* under water. If B > W then $R_b = B - W$ is the *remaining boost*. For example, if W = 30N and B = 20N, then the object under water weighs 30N – 20N = 10N. However, if W = 30N and B = 40N, then the remaining boost is 40N – 30N = 10N.[26] One time we have W – B, and the next time we have B – W to calculate. We can use this one, single formula if we understand that, under certain circumstances, a negative number result is *negative weight* that is directed toward the top.

Remaining Weight W_R *= Weight W – Boost B*

$$W_R = W - B$$

Practice 6

1. Rising and falling of a hot air balloon:

a. Through warming of the air in a hot air balloon to 90-100 degrees centigrade, its boost depends upon its volume and the outside temperature. As long as the weight of the hull, gas burner, basket, instruments, ballast, and passengers is less than the boost, the balloon will rise; otherwise it falls. Falling and rising can be regulated by activating the gas burner, or releasing ballast or warm air (through a vent at the top of the balloon). Most balloons have a volume of 2500 to 3500 cubic meters. A 3000 cubic meter balloon made out of polyester has a starting weight of not quite 1000kg., with an actual weight of about 500kg. Let us assume that a hot air balloon has a boost of 980kg. and its weight is 510kg. Calculate the remaining boost. What is the maximum additional load that it can lift? Since an exact altitude can not be maintained in a hot air balloon, it moves in a kind of wavy line. If the air inside has cooled too much then it sinks, and if the burner is activated, it will rise again. During all of this the air streams in the wind carry it further so that a wavy motion is created.

   Solution: The maximum additional load is 980kg – 510kg = 470kg. One could describe the weight without additional load as -470kg.

b. In a hot air balloon an altimeter is used to measure altitude every five minutes: These are the measurements: 330m, 340m, 360m, 430m, 500m, 520m, 510m, 480m, 450m, 420m, 390m, 360m, 330m, 305m, 320m, 380m, 450m. Draw the flight path of the balloon. Choose an appropriate measure and, assume for the graph, that the wind will move the balloon the measured equivalent of 1cm every five minutes. For every five-minute interval, calculate the rising or falling of the balloon in meters. The rise should be designated with a +, and falling with a –.

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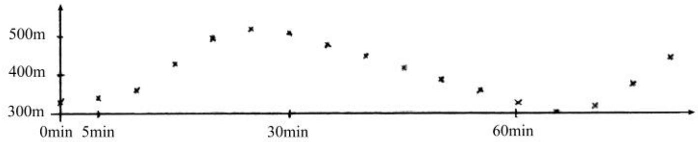
26 Teachers must decide if they wish to use the physically correct unit of measure, N (Newton), or pounds, or kilograms.

Solution: The following table shows the altitude with the differences in altitude shown below:

330		340		360		430		500		520		510		480		450	
	10		*20*		*70*		*70*		*20*		*-10*		*-30*		*-30*		*-30*

420		390		360		330		305		320		380		450
	-30		*-30*		*-30*		*-25*		*15*		*60*		*70*	



2. Rotations:
A student stands in front of the class and we ask him or her to rotate 90°. It will probably be done clockwise. But the student may also ask about the direction. We then choose one of the two directions. We now give a series of angles like 90°, 180°, 270°, or others. The student must always move the correct amount starting from the last angle given. If we now ask the student to rotate -90°, he or she would probably change their direction. Two rotations of +120° and -90° lead to the same position as one rotation of +30°. Plus and minus indicate opposing directions of rotation without a predetermination of which direction is positive and which negative. It also does not matter what the beginning position is. It is different if we arbitrarily determine a direction as a 0°- position and one rotating direction as positive. Then, using the degree of angles from 0° - 360°, every position can be designated. In the first case, our attention is drawn to the process of rotation, and, in the second case, it is drawn to the spatial positions that the rotation produces. Starting with an arbitrarily chosen 0°-position, instead of designating the different positions with 0° to 360°, one can designate them with positive and negative values between 0° and 180°. Minus 180° and plus 180° designate the same position. The 0°-position and the direction of rotation are to be arbitrarily determined. If we describe a +30°-rotation with only the rotation process we get a different aspect of positive and negative then if a fixed 0°-position is used. These differences should be brought clearly into one's consciousness.

3. Geographic Coordinates

We use similar angle systems to determine specific places on the earth and in nautical navigation. The equator provides an astronomically highlighted zero-position for latitude. Instead of positive and negative, we speak of northern and southern latitudes. Frankfurt, Germany lies just about at 50 degrees northern latitude, and the North Pole is at 90° northern latitude. The Tropic of Cancer and the Tropic of Capricorn are both at 23.5° northern and southern latitude, respectively. One finds mostly desert in these areas.
For the east-west positions, one must arbitrarily determine a zero-meridian. It runs through Greenwich, which in earlier times was the home of the most important observatory in the British Empire. The longitude of any place is given in the number of degrees east or west of Greenwich.

Geographic Data for Various Cities[1]

North America	Country	Latitude	Longitude
Mexico City	Mexico	19°26'N	99°07'W
Toronto	Canada	43°42'N	79°24'W
Sacramento, CA	USA	38°35'N	121°30'W
San Francisco, CA	USA	37°37'N	122°22'W
New York, NY	USA	40°45'N	73°59'W
New Orleans, LA	USA	29°57'N	90°04'W
South America			
Rio de Janeiro	Brasilia	22°54'S	43°13'W
Buenos Aires	Argentine	34°40'S	58°28'W
Lima	Peru	12°06'S	77°03'W
Asia			
Teheran	Iran	35°40'N	51°26'O
Bagdad	Iraq	33°12'N	44°12'O
Jerusalem	Israel	31°47'N	35°13'O
Tokyo	Japan	35°45'N	139°35'O
Ulan Bator	Mongolia	47°55'N	106°55'O
Kabul	Afghanistan	34°30'N	69°10'O
New Delhi	India	28°38'N	77 °12'O
Jakarta	Indonesia	6°08'S	106°45'O
Europe			
Sankt Petersburg	Russia	59°55'N	30°15'O
Paris	France	48°50'N	2°20'O
London	GB	51°30'N	0°00'W
Vienna	Austria	48°12'N	16°12'O
Stockholm	Sweden	59°21'N	18°05'O
Berlin	Germany	52°31'N	13°25'O
Frankfurt	Germany	50°07’N	8°40’O
Australia			
Melbourne	Australia	37°49'S	144°58'O
Africa			
Johannesburg	South Africa	26°10'S	28°02'O
Cairo	Egypt	30°03'N	31°15'O
Lagos	Nigeria	6°27'N	3°28'O
Nairobi	Kenya	1°17'S	36°50'O

Practice 7

1. There is a flight that goes from Frankfurt to Tokyo. What is the difference in degrees of latitude and longitude between these two places? Why does the flight path go so far north?

2. Search in the atlas for places with approximately the same latitude as your home. Do these places have similar climates?

Using a world globe and calculating flights going east as positive and flights going west as negative, and likewise those going north as positive and south as negative, we can start from any place and locate many other places by giving their coordinates relative to the starting point.

Example: What place is at +5°38' (north) and +29°02' (east) from Frankfurt, Germany?

(Answer: Moscow)

Movement along a line of latitude creates very different changes as those along a line of

[1] Quellen http://www.g-o.de/index08.htm und www.calsky.com

longitude. The most important reasons for this are astronomical changes. If we move towards the *Equator* then the *North Star* sinks more and more. At the Equator the North Star is on the horizon.

Therefore, the sun's ecliptic rises higher and higher until it stands in the north. South of the southern tropic (Tropic of Capricorn) it is always in the north. Between the two tropics it rhythmically moves to its zenith like a pendulum. Together with other geographic conditions, such as mountains, rivers, oceans, etc., this determines climactic changes. If an airplane flies along a line of longitude then one does not need to re-set one's watch because one stays within the same time zone. When moving along a line of latitude the astronomical conditions remain essentially the same.

Climactic changes are, above all, caused by ocean currents (Gulf Stream, Labrador Stream, etc.). However, the conditions of time are also changed. If one travels west then the day becomes longer, and, if one travels east, toward the Sun, the day becomes shorter. Then one must re-set a watch accordingly. One is dealing with changes in *time.*

4. Warming and Cooling

In normal measuring of temperatures we can again start with two opposing processes; *warming* and *cooling.* Human beings perceive both aspects with their sense of warmth and react to changes with their blood vessels and much more. If we leave a warm room and go outside into a cold winter's night our organism must react differently than if we enter a warm room. In this case, the *temperature difference* is more important than the absolute temperature. With +30°C or -30°C we can characterize a certain warming or cooling. As a matter of fact, many other factors play a role in warmth perception, such as humidity, air pressure, wind speed, and the radiant heat of the surroundings. In the middle of the constantly changing environmental conditions, the organism maintains a wonderfully constant internal temperature that is around +37°C. The organism has to constantly regulate its own temperature according to all the different environmental changes. For this reason, perceiving the changes is more important than recognizing an always defined absolute temperature, and is usually crucial for physical processes. In general, the mercury in a thermometer shows us the *changes in temperature* by its lengthening or contracting. Only when we arbitrarily define a zero point can we assign a fixed number to the temperature. The existence of the various temperature scales (Celsius, Fahrenheit, Reaumur, and Kelvin) shows that very different conventions are possible.

Practice 8

Following are the high and low temperatures for a series of days taken every twenty-four hours in late Autumn: 20°C, 14°C, 18°C, 14°C, 20°C, 8°C, 16°, 4°C, 10°C, -2°C, 7°C, -3°C. Describe the probable weather (clouds, wind direction, moisture). Make a drawing representing the temperature, and for each day, give the maximum temperature change (warming = positive, cooling = negative).

Solution: The temperature changes are as follows: -6°, +4°, -4°, +6°, -12°, +8°, -12°, +9°,-10°.

First, it is late-summer warm with a southern wind that later turns to the east or northeast. The air pressure is high. The sky is clear at night so that it is cold.

5. Moving Forward and Backward

If we move forward or backward in a straight line we can count our steps. In order to express the difference between the opposing directions we can designate going forward five steps as +5, for example, and going backward five steps as -5. In this case, plus and minus indicate the opposing directions of movement. In certain respects, what we designate as + or –

is arbitrary, even if we experience stepping forward or backward, or rising and falling from the previous example, very differently.

Mark a point O (from the word origo = origin) on a line g and determine one direction to be positive. Every point of equal distance on the line can be assigned a number: Starting with O and going five steps, or increments, to point P, is designated +5. Starting with O and going five steps backwards to point Q will be designated -5. The pure number, that is, its value, tells us how many steps from O we have gone, and the prefix sign tells us which direction from O the point lies; if we have gone forwards or backwards.

In all the above examples we have used positive and negative numbers in practical situations in *two* different aspects. In the *first* aspect, we characterized *processes*; warming and cooling to a certain degree of temperature, positive or negative weight in water, rising and falling, rotating in opposing directions, and moving forward and backward. In addition, the introduction of negative numbers included receiving and spending money. In the *second* aspect, we have looked at a *zero point* to which the highlighted changes have been related. Then the positive and negative numbers no longer characterize primarily processes, but rather *conditions* or places.

Now, one must determine if such a zero point lies in the nature of a thing or if it must be arbitrarily fixed. Looked at as a process, the zero signifies the preservation of a condition. No change takes place. Looking at the conditions, the zero can represent the transition into changed areas of quality. So it is, for example, in the transition from credit into debt. Having *nothing* is an absolute zero point. *Less than nothing* represents a completely different quality from positive assets. In measuring temperatures (Fahrenheit and Celsius), we normally use an arbitrarily chosen zero point. Only the Kelvin scale goes back to the absolute zero point of -273.15°C. Negative temperatures are not known in the Kelvin scale.[28] With forces of rising and falling, an absolute zero point is reached with the condition of suspension. Plus and minus signifies only the direction of the opposing force.

The most extrinsic relationship between numbers and an area of application, we find with *number scales*, where the zero point does not characterize any transition of quality between both sides. This very external relationship between numbers and space, that shows nothing of the quality of a thing, is what gives the number scale such a variety of uses. However, at the same time, it carries within it the danger that the thing itself is only spatially represented and so remains essentially unknown. A temperature is something other than a number on a linear scale. One comes closer to the phenomenon of warmth if one pays attention to the processes of warming and cooling.

Practice 9

1. Search for examples of polarities to which positive and negative numbers can be applied. See if there is an absolute zero point or not. Could one also apply it to the wind?
2. A Roman citizen X died at the end of the year 42 A.D. (after the birth of Christ) at just about exactly age 70. When was he born?

Historians count *without* a year zero:

3 B.C.	2 B.C	1 B.C	1 A.D	2 A.D.	3 A.D.

Astronomers count *with* a year zero. From the year 1 B.C. back, there is a difference of one year to historical calculations:

-2	-1	0	1	2	3

[28] I am saying this with caution since modern physics has already introduced many things that were originally theoretically impossible.

The year 3 (astronomically) is the third year A.D. The year zero (astronomically) is the first year B.C., and the year 1 (astronomically) is the second year B.C. The historical and astronomical time calculations before Christ differ by one year because in historical reckoning there is no year zero. This plays a role in understanding the change of the last millennium because only on 12-31-2000 was there actually 2000 years since the birth of Christ. If historians also counted a year zero, then 2000 years would have gone by already on 12-31-1999. Nevertheless, the millennium celebrations took place at the transition from 1999 to 2000 because that is when the numbers showed a new millennium.

Solutions:
1. Examples of such polarities are: Pressure and drag on a beam (absolute zero point), credit and debt (absolute zero point), elevation (no absolute zero point). Negative wind speed does not make any sense because there are no opposing pairs of wind directions.
2. 39 B.C.

I.5 More Exercises about Negative Numbers

Practice 10

Working with negative numbers is not limited to the area of *whole* numbers, but can also include fractions, or, as one also says, *rational* numbers.[29] This is especially apparent when working with money.
Calculate:

1. $3.5 - 2.3 =$
2. $3.5 - (-2.3) =$
3. $4.4 - 3.3 + (-2.2) - (-1.1) =$
4. $6.26 - 4.82 + 5.36 =$
5. $17.28 - 19.28 + 14.50 =$
6. $19.28 - 17.28 - 14.50 =$
7. $19a - 38a =$
8. $1.9b - 3.8b =$
9. $0.19c - 0.38c =$
10. $14.67 - 19.67 =$
11. $14.67 - 19.68 =$
12. $14.69 - 19.68 =$
13. $14.67 - 19.685 =$
14. $14.675 - 19.67 =$
15. $1111.111 - 2222.222 + 3333.333 =$
16. $1111.111 - 2222.222 + 3333.333 - 4444.444 =$
17. $17 - 23.45 - 182.44 + 71.223 + 11 - 23.2323 =$

[29] From the Latin ratio meaning measure or reason.

Solutions:
1. *1.2*; 2. *5.8*; 3. *0*; 4. *6.8*; 5. *12.5*; 6. *–12.5*; 7. *–19a*; 8. *–1.9b*; 9. *–0.19c*; 10. *–5*; 11. *–5.01*; 12. *–4.99*; 13. *–5.015*; 14. *–4.995*; 15. *2222.222*; 16. *–2222.222*; 17. *–129.8993*

Practice 11

1. $\frac{1}{3}-\frac{1}{2}=$ 2. $\frac{1}{3}-\frac{1}{4}=$ 3. $\frac{1}{5}-\frac{1}{4}=$

4. $\frac{a}{3}-\frac{a}{2}=$ 5. $\frac{a}{3}-\frac{a}{4}=$ 6. $\frac{a}{5}-\frac{a}{4}=$

7. $\frac{3}{a}-\frac{2}{a}=$ 8. $\frac{3}{a}-\frac{4}{a}=$ 9. $\frac{5}{a}-\frac{4}{a}=$

10. $\frac{2}{b}-\frac{3}{2b}=$ 11. $\frac{4}{3b}-\frac{3}{2b}=$ 12. $\frac{a}{b}-\frac{b}{a}=$

Solutions:
1. $-\frac{1}{6}$; 2. $\frac{1}{12}$; 3. $-\frac{1}{20}$; 4. $-\frac{a}{6}$; 5. $\frac{a}{12}$;6. $-\frac{a}{20}$; 7. $\frac{1}{a}$; 8. $-\frac{1}{a}$; 9. $\frac{1}{a}$; 10. $\frac{1}{2b}$; 11. $-\frac{1}{6b}$; 12. $\frac{a \cdot a - b \cdot b}{ab}$

Before we get involved with multiplication and division in the area of positive and negative numbers – and zero - , we will turn our attention to the basic principles of algebra.