# I Algebraic Skills I

## *I.1 Introduction*

In this chapter we begin with actual algebra. Those of you who studied the 6th grade book on Algebra (and economics) are well prepared for this chapter. Following Rudolf Steiner's suggestion, we made use of the so-called letter arithmetic to calculate interest. We will remember that we used letters as *place holders* for a given number. For some students it will be necessary to repeat the explanation for this usage because the question may arise again: How can one calculate with letters? Of course, one cannot. The reason they are used in mathematics can be explained using an example of a basic principle of algebra[1].

Just as we have been concerned with numbers and their relationships through mathematical operations in elementary arithmetic, so it is that the basic objective of algebra is to convert the *sequence of operations.* Such conversions are interesting because they lead to the same results regardless of the previous operations. Students have been using the algebraic rules that apply here for a long time. What we wish to achieve now is a comprehensive understanding. I will begin with the distributive property even though it is not the logical choice. I would like to use it as an example to show how such a property, normally found to be self-evident and without impact, can be shown in the classroom to have a very interesting quality.

## *I.2 The Basic Rules and Properties of Algebra*

### *I.2.1 The Distributive Property*

Let us start with a sum $35 + 63$ that can be simply calculated:

$$35 + 63 = 98$$

Now, let us look at each summand with a “multipliers gaze”; that is, we ask ourselves where we can form products. We get:

$$98 = 35 + 63 = 5 \cdot 7 + 9 \cdot 7$$

So, we have to add *5 sevens* and *9 sevens* together. In addition, we can count the *sevens*. It is $5 + 9 = 14$ *sevens*. This restructuring can be written out in steps by using parentheses. Whether the students have knowledge of this or not from previous classes, it is still beneficial to go over it again carefully. We use parentheses to establish the sequence of operations: The operations within the parentheses should be completed first:

$$98 = 35 + 63 = 5 \cdot 7 + 9 \cdot 7 = (5 + 9) \cdot 7 = 14 \cdot 7 = 98$$

The result is worth discussing! At first glance, it appears to be a meaningless equality or a

[1] See Chapter I.2 of this booklet

tautology. Compare only the first and last numbers in the equation and we see that it says

$$98 = 98.$$

But actually, the statements are very different: 98 = 35 + 63 represents the number 98 as the *sum* of two numbers, in this case, 35 and 63. Such a representation is possible in many different ways. For example:

$$98 = 35 + 63 = 36 + 62 = 37 + 61...$$

In $98 = 14 \cdot 7$, $98$ appears as a *product* of two numbers. $98 = 2 \cdot 49$ or $98 = 2 \cdot 7 \cdot 7$ is also possible. With that, the possibilities (in the realm of natural numbers) are already exhausted.

If the number was *97* instead of *98*, the addition breakdowns remain almost unchanged:

$$97 = 35 + 62 = 36 + 61 = 37 + 60...$$

A multiplication breakdown is not even possible! 97 is a prime number.

Where was the decisive step from the sum to the product? It was the transition:

$$5 \cdot 7 + 9 \cdot 7 = (5 + 9) \cdot 7$$

It was possible because we had the *same* factor in both summands. With the other factors we counted how often they occurred. Converting a sum into a product is called *factoring*.

For other examples, we can use the following addition breakdowns as well as many others:

$$51 = 21 + 30 = 7 \cdot 3 + 10 \cdot 3 = (7 + 10) \cdot 3 = 17 \cdot 3 = 51$$

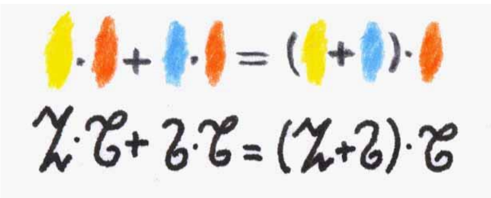
***Practice 1***

Tranform according to the given examples:

**1.** $65 = 40 + 25$; **2.** $77 = 42 + 35$; 3. $114 = 66 + 48$; 4. $24 = 18 + 6$; $24 = 16 + 8$; $24 = 10 + 14$; $24 = 12 + 12$;...

The important thing is that we have at least *one* common factor (or divider) in both summands.

If we tried to express this as a general rule we would not be able to use any numbers. Every example using numbers just presents another special case. So, we must find other means of representing, such as colors or forms. We must adhere to one rule: Within one calculation the same colors or forms must always be used to represent the same numbers. Bearing that in mind, we can express the property in the following way:



Much joyful imagination can be brought into this. Who can think of new symbols? Which ones are interesting, easy to write, and have a nice form? What problems arise when one uses self-made forms? This becomes immediately apparent when one asks another student to decipher the lines. The forms have no names. Such apparently trivial things can make the somewhat abstract subject of algebra suddenly interesting for many students.

Since letter forms are already known everywhere, have names, and are easily available for printing, we also use them for algebra, even though letters cannot be added or multiplied. This must be stated very clearly because it really is a difficult concept for some students. This is what the distributive property looks like when expressed with letters:

$$a \cdot c + b \cdot c = (a + b) \cdot c$$

The distributive property describes when a sum can be converted into a product. This is always possible when all the summands have a common factor. The common factor is c which we can "factor out" of the two summands.

1. *Note:* We introduced the use of letters in mathematics in the sixth grade. They had concrete significance: They were *abbreviations for words* such as capital, interest, etc. Taking the step into general use of letters should be discussed many times. One can introduce the term *place holder* because they mark the place of any number.

*Note:* It seems to me to be essential that one go the way from sum to product in the suggested manner. The other way round, from product to sum by multiplying out the parentheses, makes the rule almost trivial. By using parentheses, the transition to a higher operation is found, and that is often not even possible. If it is found, as in the solution to the quadratic equation through factoring, for example, then that is often the only way a solution is even possible.

At this point one should give a series of exercises using parentheses and factorization of algebraic expressions. Then, we will go into what appears to be the much easier to understand commutative and associative properties, and then have another look at the distributive property.

### *I.3 Using parentheses, brackets, and braces*

As has already been said, parentheses establish the *sequence of mathematical operations*: Operations within parentheses are to be completed first, before those either in front or behind the parentheses.

However, the order of operations must not always be determined by parentheses. In many cases the rule of *dot before line* saves having to use parentheses. For example, $3 \cdot 4 + 5$, according to the rule, must first be multiplied and then added:

$$3 \cdot 4 + 5 = 12 + 5 = 17$$

If we wish to do the addition first, then we must use parentheses:

$$3 \cdot (4 + 5) = 3 \cdot 9 = 27$$

The same applies to subtraction and division. Example:

$$16 - 12 : 4 = 16 - 3 = 13$$

As opposed to:

$$(16 - 12) : 4 = 4 : 4 = 1$$

Division is treated the same as multiplication, even though it is represented by a different symbol! It has been agreed upon that the fraction line works the same as parentheses. Therefore, the last example, written as a fraction, does not need parentheses.

$$\frac{16-12}{4} = \frac{4}{4} = 1$$

If a term in parentheses is to be added, then, according to the associative property, that we will discuss soon, the parentheses can be left out. If a term in parentheses is to be subtracted, then, as we have seen, the subtraction can be substituted with the addition of the reverse number. For this however, all addition and subtraction signs in the parentheses must be reversed. It is also expressed like this:

If there is a + sign in front of the parentheses, then they can be left out. If there is a – sign in front of the parentheses then canceling them out means that all the numbers within the parentheses have their signs reversed. The opposite is also true: If a sum or a difference is to be subtracted then it must first be put into parentheses. For instance, if we wish to subtract 2 – 3 from 4 then we need parentheses:

$4 - (2 - 3),$

We can calculate this expression in two ways:

$4 - (2 - 3) = 4 - (-1) = 4 + 1 = 5$ or $4 - (2 - 3) = 4 - 2 + 3 = 5.$

If there is a minus sign in front of parentheses in an algebraic expression then the reverse operation must be performed to cancel out the parentheses. For example:

$3a - (5a - 3a)$

is calculated:

$3a - (5a - 3a) = 3a - 2a = a$ or $3a - (5a - 3a) = 3a - 5a + 3a = a.$

This is correct for every number a.

We have to calculate:

$3a - (3a - 5a)$

And we do it like this:

$3a - (3a - 5a) = 3a - (-2a) = 3a + 2a = 5a$ or $3a - (3a - 5a) = 3a - 3a + 5a = 5a.$

Bracket expressions also can be more complicated than the examples thus far. There can be multiple brackets or brackets within brackets. In that case we speak of different *layers* of brackets. **(Trans. Note: The German word Klammernebenen can mean layers or planes of brackets)**

First, one uses the normal parentheses style brackets and when more are needed one uses the square brackets and the next level of brackets used are the curvy brackets.

In English each of these bracket styles has a name: ( ) = parentheses; [ ] = brackets; { }= braces

The previously stated rule: *Fraction lines work like brackets.*

One must be especially attentive to this when fractions occur as factors. Here are two examples:

$$8 \cdot \frac{12-8}{2} = 4 \cdot (12-8) = 4 \cdot 4 = 16 \qquad oder \qquad \frac{a+b}{2} \cdot \frac{a-b}{2} = \frac{(a+b)(a-b)}{4} = \frac{a^2-b^2}{4}$$

Pay close attention to where brackets are necessary.

*I.3.1 Examples Using Multiple Brackets*

Beginning with the inner parentheses, we calculate each term in steps. We must remember that a negative number is subtracted by adding the reverse positive number. In the first example we will keep the original brackets at first and then reduce them a level, step by step, when fewer levels are necessary:

$110 - [3 - (4 - 6)] = 10 - [3 - (-2)] = 10 - [3 + 2] = 10 - 5 = 5$

When reduced, we get the following:

$110 - [3 - (4 - 6)] = 10 - [3 - (-2)] = 10 - (3 + 2) = 10 - 5 = 5$

2. Example $2\,\{9 - 3\cdot[7 - 3\,(3 - 1) + 4] - 2\} = 2\cdot\{9 - 3\cdot[7 - 3\cdot2 + 4] - 2\} =$

$2\cdot\{9 - 3\cdot5 - 2\} = 2\cdot\{-8\} = -16$

Or, with the brackets reduced:

$2\{9 - 3\cdot[7 - 3(3 - 1) + 4] - 2\} = 2\cdot[9 - 3\cdot(7 - 3\cdot 2 + 4) - 2] =$

$2\cdot(9 - 3\cdot 5 - 2) = 2\cdot(-8) = -16$

3. $3\cdot\{10 - [9 - 8 - (7 - 6)]\} = 3\cdot[10 - (9 - 8 - 1)] = 3\cdot(10 - 0) = 30$

**4.** $5 - \frac{7-9}{2} = 5 - (-\frac{2}{2}) = 5 + \frac{2}{2} = 5 + 1 = 6$

***Practice 12***

Calculate:

1. $1 - 2 + 3$
2. $2 - 3 + 4$
3. $3 - 4 + 5$
4. $4 - 5 + 6$
5. $5 - 6 + 7$
6. $a - (a + 1) + (a + 2)$ Use this formula for the previous problems.
7. $-6 + 7 - 8$
8. $-7 + 8 - 9$
9. $-8 + 9 - 10$

What is the rule for the previous three problems?

10. $2a - 4a + 6a$
11. $4a - 6a + 8a$
12. $6a - 8a + 10a$
13. $-8a + 10a - 12a$
14. $9a - 4 - 5a$
15. $13a - 9 - 12a + 10$
16. $27b - 17 - 7b + 7$
17. $28c - 16 - 14c + 9$

Solutions: 1. 2; 2. 3; 3. 4; 4. 5; 5. 6; 6. a + 1; 7. –7; 8. –8; 9. –9; 10. 4a; 11. 6a; 12. 8a; 13. –10a; 14. 4a – 4 = 4(a – 1); 15. a + 1; 16. 20b – 10 = 10(2b – 1); 17. 14c – 7 = 7(2c – 1)

***Practice 13***

Order, combine, and reduce brackets – when possible.

1. $12a - 2b - 8a + 6b$
2. $111a - 121c - 11a + 71c$
3. $1001b + 59d - 143b + 513d$
4. $19r + 37s + 14t - 12s + 11t + 31r$
5. $2m + 3n + 4p$
6. $18a - 7 + 9a - 2$
7. $81x - 91 + 19x - 9$
8. $321y + 16x - 24 - 322y - 17x + 25$
9. $-18 + 19 - 20$
10. $-18a + 19b - 20c$
11. $-18a + 19a - 20a$
12. $-18a + 19a + 20$
13. $-18b + 19a + 20b$

Solutions: 1. $4a + 4b = 4\cdot(a + b)$; 2. $100a - 50c = 50 \cdot (2a - c)$; 3. $858b + 572d = 286\cdot(3b$

*+ 2d);* 4. $25\cdot(2r + s + t)$; 5. $2m + 3n + 4p$; 6. $9\cdot(3a - 1)$; 7. $100\cdot(x - 1)$; 8. $-y - x + 1$; 9. $-19$;
10. $-18a + 19b - 20c$; 11. $-19a$; 12. $a + 20$; 13. $19a + 2b$.

***Practice 14***

1. $(6 - 8) - (10 - 12)$
2. $(9 - 6) - (117 - 113)$
3. $(-5 + 8) + (-8 + 5)$
4. $\frac{1}{2}\cdot(4+8) - \frac{1}{3}\cdot(42-21)$
5. $\frac{1}{3}\cdot(30-6) - \frac{1}{5}\cdot(117-72)$
6. $\frac{1}{17}\cdot(50-16) - \frac{1}{11}\cdot(455-422)$
7. $2\cdot(17,2 - 14,7) - 5\cdot(28,8 - 24,6)$
8. $2,2\cdot(21 - 16) - 4\cdot(18,7 - 9,2)$
9. $2,3\cdot(3,1 - 2,9) - 3,7\cdot(4,8 - 4,7)$
10. $2a - (a - 2)$
11. $7t - (5t + 2)$
12. $(100 - 12a) - (88 - 24a)$
13. $(77 + 18x) - (88 + 18x)$
14. $(9a - 7b + 8c) - (9c - 8b + 10a)$
15. $(3a - 4b + 5c - 6d) + (6d - 5c + 4b - 3a)$
16. $(0,5a + 1,5b) - (0,5b - 1,5a)$
17. $(-1,8x - 9,2y + 12,1) - (11,1 - 2,8x - 8,2y)$
18. $(4,5r + 9,1s + 6,7t - 1,1) - (6,6t - 1 + 9s + 4,4r)$

Solutions:

1. $0$; 2. $-1$; 3. $0$; 4. $-1$; 5. $-1$; 6. $-1$; 7. $-16$; 8. $-27$; 9. $0,09$; 10. $a + 2$; 11. $2t - 2$; 12. $12\cdot(1 + a)$; 13. $-11$; 14. $-a + b - c$; 15. $0$; 16. $2a + b$; 17. $x-y+1$ ; 18. $0,1\cdot(r + s + t - 1)$.

***Practice 15***

Add or subtract:

1. $+\begin{cases} 36x+24 \\ \underline{-9x+12} \end{cases}$

2. $+\begin{cases} 72x+51 \\ \underline{-38x-51} \end{cases}$

3. $+\begin{cases} -99x+101 \\ \underline{199x-\ \ 1} \end{cases}$

4. $-\begin{cases} 36x+24 \\ \underline{-9x+12} \end{cases}$

5. $-\begin{cases} 72x+51 \\ \underline{38x-51} \end{cases}$

6. $-\begin{cases} -99x+101 \\ \underline{199x-\ \ 1} \end{cases}$

7. $+\begin{cases} 28x-35y+56z+14 \\ \underline{-21x+28y-49z-21} \end{cases}$

Solutions:

1. *27x + 36*; 2. *34x*; 3. *100·(x + 1)*; 4. *45x + 12*; 5. *34x + 102*; 6. *–298x + 102*; 7. *7·(x – y + z – 1)*.

***Practice 16***

1. $2\cdot[3-4\cdot(5-4)+3]$
2. $5\cdot[(4-3)\cdot 2+4]$
3. $3\cdot[2\cdot(5-3)-(6-4)]$
4. $2\cdot(3{,}5x-1{,}5x)-[7{,}4x-2\cdot(3{,}2x-2x)]$
5. $y\cdot(18{,}7-2{,}2)-3y\cdot(26{,}5-21)$
6. $(11-10)\{1-2+3\cdot[(4-5)-(6-7)]-8+9\}$
7. $\frac{3}{2}\cdot\left\{2\cdot\left[\left(\frac{1}{2}-\frac{1}{3}\right)-\left(4-\frac{25}{6}\right)\right]+\frac{1}{3}\right\}$
8. $\frac{8}{3}\cdot\left\{\frac{9}{2}\cdot\left[\left(\frac{1}{3}-\frac{1}{9}\right)-\left(\frac{28}{9}-3\right)\right]+\frac{1}{4}\right\}$
9. $\frac{1}{2}\cdot\left\{\frac{21}{8}\cdot\left[\left(\frac{5}{7}-\frac{16}{21}\right)+2\cdot\left(\frac{4}{7}-\frac{5}{14}\right)\right]+3\right\}$

Solutions: 1. 4; 2. 30; 3. 6; 4. -x; 5. 0; 6. 0; 7. $3/2$; 8. 2; 9. 2.

### *I.4 The Commutative Property*

A property that is often used is the commutative property. If we divide a number into two summands, n = a + b, then n = b + a is also possible.

Think of something like a tree trunk that is cut in half, or a given amount of water, then a + b or

b + a are the same thing just in a different sequence. But a + b always equals b + a.

But, if one starts with something other than the dividing of the sum of a + b, then the commutative property is not as self-evident as it first appears. In real life a + b and b + a are certainly not always the same. One need only consider two people and a = 1 dollar and b = 1 million dollars. In the first instance, one person who has only 1 dollar gets 1 million dollars; in the second instance, the millionaire gets only one dollar. The result is the same in both cases, but the process, which certainly influences the feelings of the people involved in different ways, is different.

The students can easily come up with other examples where the commutative property does not work; painting, for instance, where the result depends upon the sequence of colors; or a conversation. Examples will vary depending on who brings up the first idea. And, naturally, in processes of time there is hardly any real commutability.

The equal sign in a + b = b + a does not express equality in the sense that it is representing the same thing, but rather equality as regarding the result; when the same number can be inserted left and right for a and b respectively.

We also know of such partial equalities in real life: Two people have the same religion, or sex, etc. That is not to say they are alike in other ways. That is also how the mathematical equal sign should be understood. Here, it expresses the equality of the results on both sides, regardless of the numbers chosen for a and b.30

The commutative property also does not generally apply to multiplication in practical life. For instance, if someone needs three 5meter long beams to build a roof they cannot substitute five 3meter long beams, even though the total length remains the same.

$$3 \cdot 5m = 5 \cdot 3m = 15m$$

Again, the commutative property of the factors in a product is only applicable to the result, not the process, of the operation of multiplication. If we are concerned only with the result then we can use the commutative property for multiplication:

$$a \cdot b = b \cdot a.$$

Just as with the distributive property, we can represent the commutative property using colors or forms.

The commutative property can be applied to addition and multiplication only with the said limitations. Soon, however, we will learn about operations where it no longer applies; powers, roots, and logarithms.

### *I.5 The Associative Property*

Another rule of algebra that we have used many times says that we can add or multiply numbers in any sequence and always achieve the same result. To put it more simply and exactly, what we mean is this:

When a, b, and c are any three numbers, then

$$(a + b) + c = a + (b + c).$$

If we apply the commutative property to this then we see that all of the following

expressions result in the same value regardless of which numbers we choose for a, b, and c.

| | | | |
|---|---|---|---|
| $(a + b) + c$ | $a + (b + c)$ | $(b + c) + a$ | $b + (c + a)$ |
| $(a + c) + b$ | $a + (c + b)$ | $(c + a) + b$ | $c + (a + b)$ |
| $(b + a) + c$ | $b + (a + c)$ | $(c + b) + a$ | $c + (b + a)$ |

The same applies when a, b, or c are themselves sums of other numbers; that is, when we add more than three numbers.

Assignment: Choose numbers for a, b, and c, and test the equality of all the expressions.

### *Practice 17*

Calculate the following sums using mental arithmetic:

$3 + 4 + 6 + 7$

$3 + 16 + 17 + 4$

$3 + 16 + 17 + 14$

30 The equality expressed by “=” has the following characteristics: 1. A = A (reflexivity), 2. If A = B, then B = A (symmetry), and 3. If A = B and B = C, then A = C (transitivity). (For more meanings of the “=” see Louis Locher-Ernst, *Arithmetik und Algebra*, Dornach 1984, P. 407 and following.)

*2 + 9 + 8 + 1*

*12 + 9 + 8 + 11*

*12 + 19 + 18 + 11*

*5 + 3 + 5 + 7*

*5 + 13 + 15 + 7*

*15 + 13 + 15 + 17*

*5 + 4 + 6 + 5 + 7*

*15 + 4 + 16 + 5 + 7*

*15 + 14 + 16 + 15 + 7*

*3 + 9 + 7 + 1 + 8*

*13 + 19 + 7 + 1 + 8*

*13 + 19 + 17 + 11 + 8*

*8 + 2 + 4 + 6 + 12*

*8 + 12 + 4 + 16 + 12*

*18 + 12 + 14 + 16 + 12*

*25 + 37 + 75 + 43*

*24 + 36 + 75 + 15*

*23 + 37 + 65 + 35 + 2*

*13 + 37 + 26 + 74 + 3*

*11 + 89 + 1 + 19 + 2*

*137 + 3 + 70 + 30*

*Represent the numbers 6 and 10 as the sum of the numbers 1,2,3, and 1,2,3,4, respectively, in as many ways as possible but with each number occurring only once.*

*Put the expression a + 3b + 5c + 3b + c + 5a into its simplest form.*

*Solutions:* 1. *20*; 2. *40*; 3. *50*; 4. *20*; 5. *40*; 6. *60*; 7. *20*; 8. *40*; 9. *60*; 10. *27*; 11. *47*; 12. *67*; 13. *28*; 14. *48*; 15. *68*; 16. *32*; 17. *52*; 18. *72*; 19. *180*; 20. *150*; 21. *162*; 22. *153*; 23. *122*; 24. *240*; 25. *6= 1+2+3=1+3+2=2+1+3=2+3+1=3+1+2=3+2+1* respectively *10=1+2+3+4= 1+2+4+3=1+3+2+4=1+3+4+2=1+4+2+3=1+4+3+2=2+1+3+4=2+1+4+3=2+3+1+4= 2+3+4+1=2+4+1+3=2+4+3+1=3+1+2+4=3+1+4+2=3+2+1+4=3+2+4+1=3+4+1+2= 3+4+2+1=4+1+2+3=4+1+3+2=4+2+1+3=4+2+3+1=4+3+1+2=4+3+2+1.*[2]

26. *6·(a+b+c).*

Just as with addition, an associative property also applies to multiplication. It says that we can multiply any amount of numbers in any sequence and always achieve the same result. In

[2] This problem anticipates a combinatorial problem.

the simplest case it means this: When a, b, and c are three numbers then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. If we include the commutative property with that then we see that all the following expressions have the same value regardless of which numbers we choose for a, b, and c.

| | | | |
|---|---|---|---|
| $(a \cdot b) \cdot c$ | $a \cdot (b \cdot c)$ | $(b \cdot c) \cdot a$ | $b \cdot (c \cdot a)$ |
| $(a \cdot c) \cdot b$ | $a \cdot (c \cdot b)$ | $(c \cdot a) \cdot b$ | $c \cdot (a \cdot b)$ |
| $(b \cdot a) \cdot c$ | $b \cdot (a \cdot c)$ | $(c \cdot b) \cdot a$ | $c \cdot (b \cdot a)$ |

Assignment: Choose three simple numbers for a, b, and c and test the equality of all the expressions.

***Practice 18***

Mental Arithmetic: Expediently calculate the following products. Remember that:

$5 \cdot 2 = 2 \cdot 5 = 10, \quad 4 \cdot 25 = 25 \cdot 4 = 100, \quad 8 \cdot 125 = 125 \cdot 8 = 1000,$

but also that:

$5 \cdot 6 = 6 \cdot 5 = 30, 4 \cdot 15 = 60$

and so on.

Of course, in the following examples it is often better to use the distributive property. But re-grouping and combining should be practiced here as shown.

Calculate, for example:

$5 \cdot 24 = 5 \cdot (2 \cdot 12) = (5 \cdot 2) \cdot 12 = 10 \cdot 12 = 120$

*or* $5 \cdot 36 = 5 \cdot (6 \cdot 6) = (5 \cdot 6) \cdot 6 = 30 \cdot 6 = 180$

*or* $5 \cdot 42 = 5 \cdot (6 \cdot 7) = (5 \cdot 6) \cdot 7 = 30 \cdot 7 = 210.$

After some practice, the intermediate steps are no longer named.

1. $4 \cdot 25$, $8 \cdot 25$ , $12 \cdot 25$, $16 \cdot 25$, $20 \cdot 25$, $32 \cdot 25$, $40 \cdot 25$, $28 \cdot 25$, $29 \cdot 25$, $30 \cdot 25$, $31 \cdot 25$ ...
2. $4 \cdot 15$, $8 \cdot 15$, $12 \cdot 15$, $16 \cdot 15$, $20 \cdot 15$, $36 \cdot 15$, $37 \cdot 15$, ...
3. $8 \cdot 125$, $16 \cdot 125$, $24 \cdot 125$, $32 \cdot 125$, $40 \cdot 125$, $41 \cdot 125$, $2 \cdot 125$, $42 \cdot 125$, $3 \cdot 125$, $43 \cdot 125$, ...
4. *Try and come up with other such problems yourself.*

Using the numbers 7, 11, and 13, try and find all possible combinations where the number 1001 is the product.

5. *Calculate in the most expedient way:* $125 \cdot 25 \cdot 5 \cdot 8 \cdot 4 \cdot 2$ *and* $125 \cdot 125 \cdot 8 \cdot 8$.
6. *Calculate in the best way* $(5 \cdot 9 \cdot 8) \cdot (4 \cdot 13\ 125.)$
7. *Calculate in the best way possible the products* $1 \cdot 2$, $1 \cdot 2 \cdot 3$, $1 \cdot 2 \cdot 3 \cdot 4$...*up to* $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7$ *These numbers play an important role in many areas of mathematics. For this reason are given a special name and their own arithmetic operator: For instance, instead of* $1 \cdot 2 \cdot 3 \cdot 4$ *one writes 4! and reads it as "four factorial,"*

*Solutions*:

1. *100; 200; 300; 400; 500; 800; 1000; 700; 725; 750; 775; ...*
2. *60; 120; 180; 240; 300; 540; 555; ...*
3. *1000; 2000; 3000; 4000; 4125; 250; 4250; 375; 4375; ...*
4. *Example:* $2 \cdot 5 = 10$. *Dann ist* $34 \cdot 5 = 17 \cdot 2 \cdot 5 = 17 \cdot 10 = 170$ *und so weiter.*

5. *1001=7·11·13=7·13·11=11·7·13=11·13·7=13·7·11=13·11·7.*
6. *125 · 25 · 5 · 8 · 4 · 2 = (125 · 8) · (25 · 5) · (4 · 2) = 1000 · (125 · 8) = 1000 · 1000 = 1000000. Other answers are possible. 125 · 125 · 8 · 8 = 1000 · 1000 = 1000000.*
7. *(5 · 9 · 8) · (4 · 13 125) = (5 · 9 · 4 · 13) · (8 · 125) = (20 · 9 · 13) · 1000 = 20 · 117 ·1000 = 2340 · 1000 = 2340000.*
8. *1 · 2 = 2; 1 · 2 · 3 = 6; 1 · 2 · 3 · 4 = 6 · 4 = 24; 1 · 2 · 3 · 4 · 5 = 24 · 5 = 120; 1 · 2 · 3 · 4 · 5 · 6 = 120 · 6 = 720; 1 · 2 · 3 · 4 · 5 · 6 · 7 = 720 · 7 = 5040. Differently expressed: 3! = 2! · 3; 4! = 3! · 4; 5! = 4! · 5 etc..*

### *I.5.1 More on the Distributive Property*

The distributive property allows for different variations and additions. We will learn about the most important of these step by step:

1. One very important aspect is inversion. This rule allows us to calculate a product as the sum of partial products in the reverse direction; as opposed to factorization, it is called multiplying out the parentheses:

$$(a + b) \cdot c = c \cdot (a + b) = c \cdot a + c \cdot b.$$

In the first equation the commutative property is applied.

We have already been using this for many years when we calculate the product of 7 · 13 in two steps, for example, in that we write 13 as 10 + 3 and create two partial products that are then added together:

$$7 \cdot 13 = 7 \cdot (10 + 3) = 7 \cdot 10 + 7 \cdot 3 = 70 + 21 = 91$$

*Calculate mentally:* 7 · 3, 7 · 13, 7 · 23, 7 · 33, 7 · 43, ...; 4 · 7, 4 · 17, 4 · 27, 4 · 37, 4 · 47, ...

2. The way to the distributive property does not always go by way of the sum, in almost the same way, we can go by the difference:

If the given difference is a · c – b · c, then the factor c (a-b) is present. That is:

$$a \cdot c - b \cdot c = (a - b) \cdot c$$

If, on the other hand, a difference is to be multiplied, then we can calculate the product as the difference of two partial products:

$$c \cdot (a - b) = c \cdot a - c \cdot b.$$

We also use this effectively in practical mathematics in that we calculate, for example:

$$7 \cdot 19 = 7 \cdot (20 - 1) = 7 \cdot 20 - 7 \cdot 1 = 140 - 7 = 133$$

Calculate mentally*: 7 · 9; 7 · 19; 7 · 29; 7 · 39; 7 · 49;...; 4 · 8; 4 · 18; 4 · 28; 4 · 38;...*

3. Instead of only two summands, we can also have more with common factors. In that case, the distributive property should be applied appropriately. If n is a common factor of any amount of summands, then n can be put in parentheses and the expression can be factored out:

$$a \cdot n + b \cdot n + c \cdot n + ... = (a + b + c + ...) \cdot n$$

The other way round, a sum is multiplied by a factor n in that one multiplies each single sum. One describes this process, as has already been said, as multiplying out:

$$n \cdot (a + b + c + ...) = n \cdot a + n \cdot b + n \cdot c + ...$$

The same thing applies when there is addition and subtraction.

Calculate mentally:

*2 · 123; 3 · 123; 4 · 123; 5 · 123;...; 2 · 234; 3 · 234; 4 · 234; 5 · 234;...*

4. Up to this point we have mostly used letters in algebra so that they took the place of any given number. But, algebraic principles also apply when we think of a combined expression instead of one letter because, in the end, when we think of the letters as replacements for numbers and complete all the operations, we again have a single number.

For example, if we expressed the commutative property for multiplication as $a \cdot b = b \cdot a$, it would also be correct if b is the sum of two numbers: $b = c + d$. In that case, $a \cdot (c + d) = (c + d) \cdot a$. We have here a product for which one factor is represented as a sum. This also holds true for all the other properties; we may replace single letters with combined letters in algebraic expressions. One way we could use this method is in changing a product of two sums into one sum.

Consider the product $(a + b) \cdot (c + d)$. The first factor in this product is $(a + b)$, and the second is

$(c + d)$. According to the distributive property, we can change this product by multiplying

each summand of the second factor by the first factor. We then get:

$$(a + b) \cdot (c + d) = (a + b) \cdot c + (a + b) \cdot d.$$

If we apply the distributive property once again to each summand, then we get:

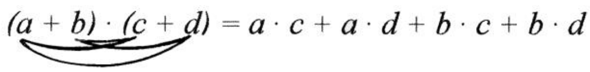
$$a \cdot c + b \cdot c + a \cdot d + b \cdot d.$$

We calculate the product of two sums by multiplying each summand of the first sum by each summand of the second sum and then adding all the products.

We could have also used the distributive property in such a way that we initially left the second parentheses unchanged. In that case, we get:

$$(a + b) \cdot (c + d) = a \cdot (c + d) = a \cdot c + a \cdot d + b \cdot c + b \cdot d.$$

These are the same summands as in the first case. This can be seen immediately by inverting the summands. It does not matter in what order the multiplication is done. However, in the classroom it is advisable to practice a fixed way at first in order to be certain that no summands are forgotten. This is my personal preference: First summand on the left, times the first summand on the right, plus the first summand on the left, times the second summand on the right, plus the second summand on the left, times the first summand on the right, plus the second summand on the left, times the second summand on the right. Below is a graphic with arcs that is something the students can remember if it is drawn carefully:




Tip: This would be a good place to point out another convention that has proven to be very practical in mathematics and already used by us: If there is no fear of misunderstanding, one can leave out the multiplication sign between a number and a letter, or between two letters. Instead of writing $2 \cdot a$, one writes 2a, or instead of $a \cdot b$, one writes ab. One can say the times if need be, or also leave it out. Of course, one may not leave out the multiplication sign between two numbers. After multiplying, often the best order is; first the numbers, then the letters alphabetically.

Our formula for multiplying simple sums looks like this:

$$(a + b)(c + d) = ac + ad + bc + bd.$$

We actually have used this formula often in earlier grades when we multiplied two-digit numbers, for instance. If we are calculating $17 \cdot 23$, for example, we can understand the two numbers as sums:

$17 \cdot 23 = (10 + 7) \cdot (20 + 3)$

If we multiply it out we get:

$$17 \cdot 23 = (10 + 7) \cdot (20 + 3) = 10 \cdot 20 + 10 \cdot 3 + 7 \cdot 3 = 200 + 30 + 140 + 21 = 391.$$

Of course, one can often calculate such a product in a more expedient way; in the above example, for instance, by not splitting the number 17 into a sum, and calculating it like this:

$$17 \cdot 23 = 17 \cdot (20 + 3) = 340 + 51 = 391$$

But with written multiplication we always did it the first way.

While multiplying out the parentheses only requires careful attention to the rules, changing the sums into products (factoring) is often an art that requires long practice before one becomes skillful. Some difficult problems, like solving quadratic equations, for example, can become really complicated, and here also, we will gain the necessary skill only through practice.

### *I.6 The Numbers Zero and One*

At the appropriate time the special role of the numbers zero and one should be looked at. They are interesting in many respects. In reality, for a long time they were not even considered actual numbers since when counting things one normally does not indicate either the single instance, or the absence of an object, with a number. When speaking of an object, a chair perhaps, and I say, “I see a chair”, then I am speaking of the existence of the chair. Only if I say, “I see one chair”, do I put it in relationship to other possible chairs, and the “one” becomes a number word. With two chairs, I recognize something common to both objects, a repetition of the same thing. With that, multiplicity appears. We are only conscious of zero as a number when our expectation and perception are not in agreement; for instance, if we expect to find chairs in a room and none are there. However, we generally do not have these kinds of experiences on a steady basis.

Regardless of the epistemology, the numbers zero and one have long found their appropriate place in mathematics. One of their most important characteristics is: If we multiply a number or expression by one, the number's value does not change. The same applies if we divide by one. With the zero: If we add zero to a number or expression, the number's value does not change. The same applies if we subtract zero. This is how the formulas look:

$$1 \cdot a = a \cdot 1 = a, a : 1 = a, 0 + a = a + 0 = a, a – 0 = a$$

An especially interesting case is when we multiply a number or expression by zero. As has already been said, if we understand a product to be a sum with equal summands, such as

$$a \cdot b = \underbrace{b + b + ... + b}_{a}$$

then it makes little sense when a = 0. How should one write a sum with zero summands?

If one looks at this sequence,

$$3 \cdot a, 2 \cdot a, 1 \cdot a, 0 \cdot a,$$

we see that each successive element of a is smaller than the one before. But it does make sense when one puts $0 \cdot a = 0$. The rule is as follows:

If a number or expression is multiplied by zero, the result is always zero.

However, what was previously possible: calculating each number in $a \cdot b = c$ from two of them – namely, b = c : a and a = c : b – is not possible if one of the numbers is zero. That has to do with the fact that the expression 0 : 0 is undefined, which means that it could have any number value. This plays an essential role in differential calculus. In the middle school one simply explains: Nothing may be divided by zero because if a : a = 1, then it would follow that 0 : 0 = 1. And, since 0 · 2 = 0, the division of both sides would lead to 2 = 1, and that is wrong.

In algebra, in the area of numbers known to us, there is another related circumstance that plays an important role: If none of the factors of a product a · b · ... · n is zero, then the product is also not zero. Or, reversed: If a product is a · b · ... · n = 0, then at least one of the factors is zero.

## *I.7 Practice in Factoring*

### *Practice 19*

Calculate the following sums. Look for common factors of the original summands, factor them out, and calculate the value again. Compare.

Example: 5 + 15. Solution: a) 5 + 15 = 20. b) 5 + 15 = 5 · (1 + 3) = 5 · 4 = 20.

Both results concur.

1. 6 + 9
2. *8 + 12*
3. *15 – 20*
4. *18 + 24*
5. *21 – 28*
6. *6 + 9 – 12*
7. *8 + 12 – 16*
8. *15 + 20 – 25*
9. *18 + 24 – 30*
10. *21 + 28 – 35*
11. *18 + 42*
12. *38 – 57*
13. *34 + 51*
14. *42 – 84*
15. *56 + 105*
16. *14 – 63 + 126*
17. *24 – 36 + 48*
18. *34 + 68 – 136*
19. *45 + 105 + 225*
20. *93 – 39 – 339*
21. *110 – 220 +330 – 440 + 550*
22. *51 + 69 + 207 – 108 – 39*

### *3.7.2 Practice 20*

Do these the same as in practice 19 (3.7.1)

Example: 0.5 + 1.5. Solution: a) 0.5 + 1.5 = 2. b) 0.5 + 1.5 = 0.5 · (1 + 3) = 0.5 · 4 = 2, or c) 0.5 + 1.5 = 5 · (0.1 + 0.3) = 5 · (0.1 + 0.3) = 5 · 0.4 = 2.

All three results concur.

1. 0,6 + 0,9
2. 0,8 + 1,2
3. 1,5 – 2

4. 1,8 + 2,4
5. 0,21 – 0,28
6. 0,6 – 1,2 –2,4
7. 0,5 + 1
8. 0,5 + 2
9. 0,5 + 0,05
10. 0,6 + 0,06 + 0,006
11. $1 + \frac{1}{2}$
12. $\frac{1}{2} + \frac{1}{4}$
13. $\frac{1}{2} - \frac{1}{4} - \frac{1}{8}$
14. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$
15. $\frac{1}{3}+\frac{2}{9}$
16. $\frac{2}{3}+\frac{5}{9}$
17. $\frac{3}{7}-\frac{6}{14}$
18. $\frac{3}{7} - \frac{5}{14}$
19. $\frac{9}{5} - \frac{18}{15}$
20. $\frac{9}{5} - \frac{17}{15}$

### *3.7.3 Practice 21*

Find as many factors as possible. Prove it by multiplying it out.

1. ad +bd
2. ab – ac
3. 2ab + 2ac
4. 2ab – 4ac
5. 2ab + 4bc
6. 4cd – 6ce
7. 7abc + 21bcd
8. 51akmt – 102bkt
9. 38cxy +85cy
10. a + a·a
11. b + 2b·b
12. 2c + 6c·c
13. 3k – 9k·k
14. am + a·a·m
15. ak – 2akm
16. abc – abcd
17. $\frac{1}{4}ab+\frac{1}{2}ac$
18. $\frac{1}{9}uv-\frac{1}{3}uvw$
19. $\frac{1}{3}xy+\frac{2}{3}xz$
20. $\frac{7}{9}rst+\frac{14}{27}rs$

In addition: Put in the given numbers and calculate the results. To check your answers, use the solutions to the previous problems and recalculate the results.

Example: ad + bd = (a + b) · d. If a=4, b=5, d=6, then on the left is: 4 · 6 + 5 · 6 = 24 + 30 = 54. On the right is: (4 + 5) · 6 = 9 · 6 = 54. Both results concur!

| | | |
|---|---|---|
| 21. ad + bd | for | a=4, b=5, d=6 |
| 22. ab – ac | for | a=2, b=7, c=5 |
| 23. 2ab + 2ac | for | a=13, b=28,3, c=-28,3 |
| 24. 2ab – 4ac | for | a=-2, b=-4, c=2 |
| 25. 2ab + 4bc | for | a=1,5, b=6, c=-3 |
| 26. 4cd – 6ce | for | c=11, d=4, e=0,5 |
| 27. 7abc + 21bcd | for | a=-1, b=-2, c=-3, d=-1 |
| 28. 51akmt – 102bkt | for | a=2, b=0,5, k=1, m=2, t=0,5 |
| 29. 38cxy +85cy | for | c=1, x=2, y= $\frac{1}{19}$ |
| 30. a + a·a | for | a=0, a=1 through a=10 |
| 31. b + 2b·b | for | b=0 through b=10 |
| 32. 2c + 6c·c | for | c=0 through c=10 |
| 33. 3k – 9k·k | for | k=0 through k=10 |
| 34. am + a·a·m | for | a=0 through a=10 |
| 35. ak – 2akm | for | a=2, $k=-\frac{1}{2}$, m=5 |
| 36. abc – abcd | for | a=b=c=2, d=-1 |
| 37. $\frac{1}{4}ab+\frac{1}{2}ac$ | for | a=8, b= $\frac{1}{2}$, c= $\frac{1}{4}$ |
| 38. $\frac{1}{9}uv-\frac{1}{3}uvw$ | for | u=3, v=6, w=1 |
| 39. $\frac{1}{3}xy+\frac{2}{3}xz$ | for | x=12, y=8, $z=\frac{1}{2}$ |
| 40. $\frac{7}{9}rst+\frac{14}{27}rs$ | for | r=3, s=9, t= $\frac{1}{3}$ |

### *3.7.4 Practice 22*

Divide, by finding as many factors as possible by removing parentheses, and dividing those factors as far as possible by the divisor.

Example: (4ac – 2bc):2c = [2c·(2a – b)] : 2c – b. Written as fractions, it looks like this:

$$\frac{4ac-2bc}{2c}=\frac{2c\cdot(2a-b)}{2c}=2a-b$$

1. (ad + bd):d
2. (ab – ac):a
3. (2ab + 2ac):2a
4. 2ab – 4ac):2a
5. (2ab + 4bc):2b
6. (4cd – 6ce):2c
7. (7abc + 21bcd):7bc
8. 51akmt – 102bkt):51kt
9. (38cxy +85cy) : 19cy
10. (a + a·a) : a
11. (b + 2b·b) : b
12. (2c + 6c·c) : 2c
13. (3k – 9k·k) : 3k
14. (am + a·a·m) : am
15. (ak – 2akm) : ak

16. (abc + bcd + cde) : c
17. (2ak + 4bk + 20ck) : 2k
18. $\frac{7uvw - 28vw + 49rvw}{-7vw}$
19. $\frac{-26r-13rs+39r\cdot r}{-13r}$
20. $\frac{4a-6ab+12ac}{-4a}$

In addition: Put in the given numbers and calculate the results. To check your answers, use the results of the previous problems and recalculate.

21. (ad + bd):d *for* a=4, b=5, d=6
22. (ab – ac):a *for* a=2, b=7, c=5
23. (2ab + 2ac):2a *for* a=13, b=28,3, c=-28,3
24. (2ab – 4ac):2a *for* a=-2, b=-4, c=2
25. (2ab + 4bc):2b *for* a=1,5, b=6, c=-3
26. (4cd – 6ce):2c *for* c=11, d=4, e=0,5
27. (7abc + 21bcd):7bc *for* a=-1, b=-2, c=-3, d=-1
28. (51akmt – 102bkt):51kt *for* a=2, b=0,5, k= $\frac{1}{19}$ =1, m=2, t=0,5
29. (38cxy +85cy) : 19cy *for* c=1, x=2, y
30. (a + a·a) : a *for* a=10
31. (b + 2b·b) : b *for* b=10
32. (2c + 6c·c) : 2c *for* c=10
33. (3k – 9k·k) : 3k *for* k=10
34. (am + a·a·m) : am *for* 10
35. (ak – 2akm) : ak *for* a=2, $k=-\frac{1}{2}$, m=5
36. (abc + bcd + cde) : c *for* a=b=c=2, d=-1
37. (2ak + 4bk + 20ck :2k *for* a=7, b=1,5, c=2, k=3
38. $\frac{7uvw - 28vw + 49rvw}{-7vw}$ *for* r=1, u=0, v= $\frac{-1}{7}$ , w=7
39. $\frac{-26r-13rs+39r\cdot r}{-13r}$ *for* $r = s = \frac{1}{13}$
40. $\frac{4a-6ab+12ac}{-4a}$ *for* $a = b = 2,\ c = \frac{1}{8}$

*3.7.5 Practice 23*

Find as many factors as possible.

1. 0,2ab – 0,4ac
2. 0,2ab – 0,5ac
3. 4ac – 0,5ac
4. 0,7abc – 1,4bc + 0,7ac
5. 2,2uv + 3,3vw – 4,4tv
6. 54uu +108uv – 135uw
7. 315 + 420 +210
8. 836x +1254xx
9. 2295a +3366ab - 2244ac
10. 1001r + 1078rs

Solutions:

Practice 19 (3.7.1):

1. 3·(2+3)=3·5=15

2. 4·(2+3)=4·5=20
3. 5·(3–4)=5·(-1)=-1
4. 6·(3+4)=6·7=42
5. 7·(3–4)=7·(-1)=-7
6. 3·(2+3–4)=3·1=3
7. 4·(2+3–4)=4·1=4
8. 5·(3+4–5)=5·2=10
9. 6·(3+4–5)=6·2=12
10. 7·(3+4–5)=7·2=14
11. 6·(3+7)=6·10=60
12. 19·(2–3)=19·(-1)=-19
13. 17·(2+3)=17·5=85
14. 42·(1–2)=42·(-1)=-42
15. 7·(8+15)=7·23)=161
16. 7·(2–9+18)=7·11=77
17. 12·(2–3+4)=12·3=36
18. 34·(1+2–4)=34·(-1)=-34
19. 15·(3+7+15)=15·25=375
20. 3·(31–13–113)=3·(-95)=-285
21. 110·(1–2+3–4+5)=110·3=330
22. 3·(17+23+69–36–13)=3·60=180

Practice 20 (3.7.2):

1. 3·(0,2+0,3)=3·0,5=1,5 oder 0,3·(2+3)=0,3·5=1,5
2. 4·(0,2+0,3)=4·0,5=2 oder 0,4·(2+3)=0,4·5=2
3. 5·(0,3–0,4)=5·(-0,1)=-0,5 oder 0,5·(3–4)=0,5·(-1)=-0,5
4. 6·(0,3+0,4)=6·0,7=4,2 oder 0,6·(3+4)=0,6·7=4,2
5. 6·(0,1–0,2–0,4)=6·(-0,5)=-3 oder 0,6·(1–2–4)=0,6·(-5)=-3
6. 5·(0,1+0,2)=5·0,3=1,5 oder 0,5·(1+2)=0,5·3=1,5
7. 5·(0,1+0,4)=5·0,5=2,5 oder 0,5·(1+4)=0,5·5=2,5
8. 5·(0,1+0,01)=5·0,11=0,55 oder 0,05·(10+1)=0,05·11=0,55
9. $1+\frac{1}{2}=\frac{1}{2}\cdot(2+1)=\frac{3}{2}$
10. $\frac{1}{2}+\frac{1}{4}=\frac{1}{4}\cdot(2+1)=\frac{3}{4}$
11. $\frac{1}{2}-\frac{1}{4}-\frac{1}{8}=\frac{1}{8}\cdot(4+2+1)=\frac{7}{8}$
12. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=\frac{1}{16}\cdot(8+4+2+1)=\frac{15}{16}$
13. $\frac{2}{5}+\frac{5}{9}=\frac{1}{3}\cdot\left(2+\frac{5}{3}\right)=\frac{1}{3}\cdot\frac{11}{3}=\frac{11}{9}$
14. $\frac{3}{7}-\frac{6}{14}=\frac{3}{7}-\frac{3}{7}=0$
15. $\frac{3}{7}-\frac{5}{14}=\frac{1}{7}\cdot\left(3-\frac{5}{2}\right)=\frac{1}{7}\cdot\frac{1}{2}=\frac{1}{14}$
16. $\frac{9}{5}-\frac{18}{15}=\frac{9}{5}-\frac{6}{5}=\frac{3}{5}$
17. 7·(0,3–0,4)=7·(-0,1)=-0,7 oder 0,7·(3–4)=0,7·(-1)=-0,7
18. 6·(0,1+0,01+0,001)=6·0,111=0,666 oder 0,006·(100+10+1)=0,006·111=0,666
19. $\frac{1}{3}+\frac{2}{9}=\frac{1}{3}\cdot\left(1+\frac{2}{3}\right)=\frac{1}{3}\cdot\frac{5}{3}=\frac{5}{9}$

20. $\frac{9}{5}-\frac{17}{15}=\frac{1}{5}\cdot\left(9-\frac{17}{3}\right)=\frac{1}{5}\cdot\frac{10}{3}=\frac{10}{15}=\frac{2}{3}$

Practice 21 (3.7.3):

1. (a + b)d
2. a(b – c)
3. 2a(b + c)
4. 2a(b – c)
5. 2b(a + c)
6. 2c(d – e)
7. 7bc(a + 3d)
8. 51kt(am – 2b)
9. 19cy(2x + 5)
10. a(1 + a)
11. b(1 + 2b)
12. 2c(1 + 3c)
13. 3k(1 – 3k)
14. am(1 + a)
15. ak(1 – 2m)
16. abc(1 – d)
17. $\frac{1}{2}a(\frac{1}{2}b+c)$
18. $\frac{1}{3}uv(\frac{1}{3}-w)$
19. $\frac{1}{3}x(y+2z)$
20. $\frac{7}{9}rs(t+\frac{2}{3})$

Practice 22 (3.7.4):

1. a + b
2. b – c
3. b + c
4. b – 2c
5. a + 2c
6. 2d – 3e
7. a + 3d
8. am – 2b
9. 2x + 5
10. 1 + a
11. 1 + 2b
12. 1 + 3c
13. 1 – 3k
14. 1 + a
15. 1 – 2m
16. ab + bd + de
17. a + 2b + 10c
18. –u + 4 –7r
19. 2 + s – 3r
20. $-1+\frac{2}{3}b-3c$

Practice 23 (3.7.5):

1. 0,2a(b – 2c)
2. a(0,2b – 0,5c)
3. 3,5ac
4. 0,7c(ab – 2b + a)
5. 1,1v(2u + 3w –4t) oder 11v(0,2u + 0,3w – 0,4t)
6. 27u(2u + 4v – 5w)
7. 105·(3 + 4 + 2) = 105 · 9 = 945
8. 418x(2 + 3x)
9. 51a(45 + 66b - 44c)
10. 77r(13 + 14s)

## *3.8 Powers*

Although we will take a look at higher mathematical operations only in a later chapter, it is appropriate that powers are introduced now as a new operation. We agree that a product with the same factors can be written in a shorter form by writing the number of factors above the same factor. This is the way it is written: 5 · 5 · 5 shortened is 53 or 2 · 2 · 2 · 2 · 2 shortened is 25 .

Generally, we write:

$$\underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n} = a^n$$

The a is referred to as the base number and the n as the exponent.

Combined expressions can also stand for a, as in this example:

$(a + b)(a + b)(a + b) = (a + b)^3$ .

The second power is also referred to as the *square (square number)* because of its use in geometry when calculating areas. One calls it *squaring* when the second power is used. In the same way, when the third power is used it is called *cube, cubic number,* and *cubing.*

***Practice 24***

Here, a series of simple exercises will be done in order to become familiar with powers. Over a long period of time, the simpler powers should be repeatedly calculated as an oral exercise. While doing so, often the mistake is made of multiplying with the exponent instead of seeing it as the *number* of equal factors. One can counteract this by speaking the following while calculating $4^3$, for example: “One times 4 is 4, times 4 is 16, times 4 is 64.”

1. Calculate as many powers of the numbers 1 – 12 as possible, and put them in the form of a table. With time, this table can be expanded and partially memorized.

2. Express these numbers using exponents: 8, 9, 64, 81, 100, 125, 343, 625, 729, 1000, 1024, 10000, 100000, 1000000.

Solutions:

**Table of First Powers**

| | | **Exponent** | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** |
| **B** | **1** | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| **a** | **2** | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1.024 |
| **s** | **3** | 3 | 9 | 27 | 81 | 243 | 729 | 2.187 | 6.561 | 19.683 | 59.049 |
| **i** | **4** | 4 | 16 | 64 | 256 | 1.024 | 4.096 | 16.384 | 65.536 | 262.144 | 1.048.576 |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| s | **5** | 5 | 25 | 125 | 625 | 3.125 | 15.625 | 78.125 | 390.625 | 1.953.125 | 9.765.625 |
| | **6** | 6 | 36 | 216 | 1.296 | 7.776 | 46.656 | 279.936 | 1.679.616 | 10.077.696 | 60.466.176 |
| | **7** | 7 | 49 | 343 | 2.401 | 16.807 | 117.649 | 823.543 | 5.764.801 | 40.353.607 | 282.475.249 |
| | **8** | 8 | 64 | 512 | 4.096 | 32.768 | 262.144 | 2.097.152 | 16.777.216 | 134.217.728 | 1.073.741.824 |
| | **9** | 9 | 81 | 729 | 6.561 | 59.049 | 531.441 | 4.782.969 | 43.046.721 | 387.420.489 | 3.486.784.401 |
| | **10** | 10 | 100 | 1.000 | 10.000 | 100.000 | 1.000.000 | 10.000.000 | 100.000.000 | 1.000.000.000 | 10.000.000.000 |
| | **11** | 11 | 121 | 1.331 | 14.641 | 161.051 | 1.771.561 | 19.487.171 | 214.358.881 | 2.357.947.691 | 25.937.424.601 |
| | **12** | 12 | 144 | 1.728 | 20.736 | 248.832 | 2.985.984 | 35.831.808 | 429.981.696 | 5.159.780.352 | 61.917.364.224 |
| | **13** | 13 | 169 | 2.197 | 28.561 | 371.293 | 4.826.809 | 62.748.517 | 815.730.721 | 10.604.499.373 | 137.858.491.849 |
| | **14** | 14 | 196 | 2.744 | 38.416 | 537.824 | 7.529.536 | 105.413.504 | 1.475.789.056 | 20.661.046.784 | 289.254.654.976 |
| | **15** | 15 | 225 | 3.375 | 50.625 | 759.375 | 11.390.625 | 170.859.375 | 2.562.890.625 | 38.443.359.375 | 576.650.390.625 |
| | **16** | 16 | 256 | 4.096 | 65.536 | 1.048.576 | 16.777.216 | 268.435.456 | 4.294.967.296 | 68.719.476.736 | 1.099.511.627.77 |
| | **17** | 17 | 289 | 4.913 | 83.521 | 1.419.857 | 24.137.569 | 410.338.673 | 6.975.757.441 | 118.587.876.497 | 2.015.993.900.44 |
| | **18** | 18 | 324 | 5.832 | 104.976 | 1.889.568 | 34.012.224 | 612.220.032 | 11.019.960.576 | 198.359.290.368 | 3.570.467.226.62 |
| | **19** | 19 | 361 | 6.859 | 130.321 | 2.476.099 | 47.045.881 | 893.871.739 | 16.983.563.041 | 322.687.697.779 | 6.131.066.257.80 |
| | **20** | 20 | 400 | 8.000 | 160.000 | 3.200.000 | 64.000.000 | 1.280.000.000 | 25.600.000.000 | 512.000.000.000 | 10.240.000.000.0 |
| | **21** | 21 | 441 | 9.261 | 194.481 | 4.084.101 | 85.766.121 | 1.801.088.541 | 37.822.859.361 | 794.280.046.581 | 16.679.880.978.2 |
| | **22** | 22 | 484 | 10.648 | 234.256 | 5.153.632 | 113.379.904 | 2.494.357.888 | 54.875.873.536 | 1.207.269.217.792 | 26.559.922.791.4 |
| | **23** | 23 | 529 | 12.167 | 279.841 | 6.436.343 | 148.035.889 | 3.404.825.447 | 78.310.985.281 | 1.801.152.661.463 | 41.426.511.213.6 |
| | **24** | 24 | 576 | 13.824 | 331.776 | 7.962.624 | 191.102.976 | 4.586.471.424 | 110.075.314.176 | 2.641.807.540.224 | 63.403.380.965.3 |
| | **25** | 25 | 625 | 15.625 | 390625 | 9.765.625 | 244140625 | 6.103.515.625 | 152.587.890.625 | 3.814697265.625 | 95.367.431.640.6 |

2. $8 = 2^3$; $9 = 3^2$; $64 = 2^6 = 4^3 = 8^2$; $81 = 3^4 = 9^2$; $100 = 10^2$; $125 = 5^3$; $343 = 7^3$; $625 = 5^4 = 25^2$; $729 = 3^6 = 9^3 = 27^2$; $1000 = 10^3$; $1024 = 2^{10} = 4^5 = 32^2$; $10000 = 10^4 = 100^2$; $100000 = 10^5$; $1000000 = 10^6 = 100^3 = 1000^2$.[3]

[3] Auf die Angabe der 1. Potenz wird hier noch verzichtet, da die erste Potenz erst später eingeführt wird. xxx